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Extremality, Symmetry and Regularity Issues in Harmonic Analysis

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Extremality, Symmetry and Regularity Issues in Harmonic Analysis

by

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To Vanessa,

for all the love and joy in this journey.

&

To Danilo and Ednir,

for everything they taught me.

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Extremality, Symmetry and Regularity Issues in Harmonic Analysis

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In this Ph.D. thesis we discuss four different problems in analysis: (a) sharp inequalities related to the restriction phenomena for the Fourier transform, with emphasis on some Strichartz-type estimates; (b) extremal approximations of exponential type for the Gaussian and for a class of even functions, with applications to analytic number theory; (c) radial symmetrization approach to convolution-like inequalities for the Boltzmann collision operator; (d) regularity of maximal operators with respect to weak derivatives and weak continuity.

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Chapter 1

Introduction

This Ph.D. thesis is composed of four independent chapters that orbit around one central theme: *Analysis*.

Chapter 2 summarizes the results of a project developed under the mentorship of William Beckner on sharp constants and maximizers for geometric inequalities related to the restriction phenomena for the Fourier transform. Most of the attention here is devoted to a sharp inequality for the Strichartz norm of solutions of the Schrödinger equation, that has connections with the restriction/extension problem for the paraboloid. The results of this chapter are collected in the paper [15].

Chapter 3 describes a project done in collaboration with Jeffrey Vaaler and more recently with Friedrich Littmann (North Dakota State University) on extremal entire approximations of prescribed exponential type for real-valued functions. The classical example of this theory is the problem of approximating $f(x) = \text{sgn}(x)$ that was considered by A. Beurling and A. Selberg in the 1930's and 1970's. In the last three years we extended this theory to a new class of even functions that include $f(x) = \log|x|$ and $f(x) = |x|^{\beta}$, $\beta > -1$, providing applications to analytic number theory and equidistribution theory. Chapter 3 gives a historical overview and presents a new extension to the theory, providing the solution of the extremal problem for the Gaussian $f(x) = e^{-\pi\lambda x^2}$, $\lambda > 0$, and its applications. These results are collected in the papers [14], [16], [18] and [19].

Chapter 4 describes the results obtained in a joint project with Ricardo Alonso, William Beckner and Irene Gamba on the use of radial symmetrization techniques to obtain sharp inequalities for operators derived from the Boltzmann equation in kinetic theory. With this machinery we can simplify many technical proofs in this theory, extend the range and obtain explicit constants for some inequalities that reveal the convolution behavior of the Boltzmann collision operator. We prove a full version of Young's inequality for this operator in the case of hard potentials, and a Hardy-Littlewood-Sobolev inequality in the case of soft potentials. The presentation in Chapter 4 is for the case of elastic collisions, but extensions of our results to the inelastic setting are also possible. These results are collected in the papers [1] and [2]. It is worth mentioning that Alonso and Gamba have recently applied these new inequalities in the paper [3] obtaining new results on the existence and regularity theory for solutions of the Boltzmann equation in the case of soft potentials.

Chapter 5 discusses new improvements on the regularity theory for maximal operators. This was a joint work with Diego Moreira (University of Iowa). The behavior of the maximal operator with respect to weak derivatives was first studied by J. Kinnunen [39] in 1997, when he proved that the classical Hardy-Littlewood maximal operator is bounded on the Sobolev space

$W^{1,p}(\mathbb{R}^n)$, for $p > 1$. In Chapter 5 we extend this theory to the bilinear maximal operator, which by a theorem of M. Lacey [43] admits L^1 as a target space, thus breaking the previous techniques that are based on the reflexivity of the spaces L^p, for $p > 1$. We overcome this difficulty by adopting a geometric measure theory approach. At the end of the chapter we also discuss the regularity of the classical maximal operator with respect to weak convergence. These results are collected in the paper [17].

Chapter 2

Sharp Strichartz Inequalities

2.1 Preliminaries

Geometric inequalities have been a topic of intense research in analysis over the last 40 years. The quest for the sharp forms of these inequalities reveals a deeper understanding of the structure of the underlying manifolds and has connections with geometric partial differential equations and Riemannian geometry. Among the most celebrated works in the last four decades, we highlight Beckner's thesis [7] on the sharp Hausdorff-Young's inequality for the Fourier transform and the sharp Young's inequality for convolutions, Lieb's work [46] on the sharp Hardy-Littlewood-Sobolev inequality, and the works of many authors in the sharp Sobolev inequalities and their connections with the Yamabe problem in differential geometry, for instance Aubin [5], Talenti [62], Escobar [23] and Beckner [8].

In this chapter we turn our attention to the problem of obtaining sharp constants and maximizers for inequalities related to the restriction phenomena for the Fourier transform. A family of examples of inequalities of this type are the Strichartz estimates for the Schrödinger equation, that are connected to the restriction problem for the paraboloid.

Let $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ be the solution of the linear Schrödinger equation

$$\left\{ \begin{array}{rcl} iu_t + \Delta u & = & 0 \\ u(0,x) & = & f(x) \,. \end{array} \right. \tag{2.1}$$

The homogeneous Strichartz estimates [20, Theorem 2.3.3] are inequalities of the type

$$\|u(t,x)\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^n)} \leq C\, \|f\|_{L^2(\mathbb{R}^n)},$$

with

$$\|u(t,x)\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^n)} = \left[\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |u(t,x)|^r \mathrm{d}x \right)^{q/r} \mathrm{d}t \right]^{1/q}.$$

The pair of exponents (q, r) is admissible if

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}\,,$$

with $2 \leq q, r \leq \infty$ and $(q, r, n) \neq (2, \infty, 2)$. The sharp forms of the Strichartz inequalities were first investigated in 2003, in a paper by Kunze [42], who showed the existence of maximizers in the case $n = 1$, $(q, r) = (6, 6)$, by concentration-compactness techniques. Later, in 2006, Foschi [26] and Hundertmark and Zharnitsky [38] independently obtained the sharp constants in the cases $n = 1$, $(q, r) = (6, 6)$; and $n = 2$, $(q, r) = (4, 4)$; showing that the only maximizers are Gaussians. They conjectured that in the case $q = r = 2+4/n$, $n \geq 3$, the extremals for the Strichartz inequalities should be given by Gaussians. In 2008, Bennett, Bez, Carbery and Hundertmark [9] applied heat-flow monotonicity techniques to offer a new proof that Gaussians yield the sharp constants in the cases $n = 1$, $(q, r) = (6, 6)$; $n = 1$, $(q, r) = (8, 4)$; and $n = 2$, $(q, r) = (4, 4)$. Also in 2008, Shao [59] showed that maximizers do exist for the non-endpoint Strichartz inequalities ($q \neq 2$ if $n \geq 3$ and $q \neq 4$ if $n = 1$) in all dimensions.

2.1.1 Main Result

In this chapter we generalize the beautiful argument of Hundertmark and Zharnitsky [38] to prove the following sharp inequality for the Strichartz norm.

Theorem 2.1.1. *Let $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ be the solution of the linear Schrödinger equation* (2.1)*. For $k \in \mathbb{Z}$, $k \geq 2$ and $(n,k) \neq (1,2)$ we have*

$$\|u(t,x)\|_{L_t^{2k} L_x^{2k}(\mathbb{R}\times\mathbb{R}^n)} \leq \left(C_{n,k} \int_{\mathbb{R}^{nk}} |\widehat{F}(\eta)|^2 \, K(\eta)^{\frac{n(k-1)-2}{2}} \, \mathrm{d}\eta \right)^{1/2k}, \tag{2.2}$$

with

$$C_{n,k} = \left[2^{n(k-1)-1} \, k^{n/2} \, \pi^{(n(k-1)-2)/2} \, \Gamma\left(\tfrac{n(k-1)}{2}\right) \right]^{-1}. \tag{2.3}$$

On the right hand side of (2.2) *we write $\eta \in \mathbb{R}^{nk}$ as $\eta = (\eta_1, \eta_2, ..., \eta_k)$ with each $\eta_i \in \mathbb{R}^n$; $F(\eta) = f(\eta_1)f(\eta_2)...f(\eta_k)$; and the kernel*

$$K(\eta) = \frac{1}{k} \sum_{1 \leq i < j \leq k} |\eta_i - \eta_j|^2.$$

This inequality is sharp and equality occurs if and only if f is a Gaussian.

Throughout this chapter we will adopt the definition of the Fourier transform of the function $f : \mathbb{R}^n \to \mathbb{C}$ given by

$$\widehat{f}(\omega) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\omega \cdot x} f(x) \, \mathrm{d}x.$$

We observe that the solution of (2.1) can be given in terms of the Fourier transform

$$u(t,x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\omega}\, e^{-it|\omega|^2}\, \widehat{f}(\omega)\, \mathrm{d}\omega. \tag{2.4}$$

The maximizers in Theorem 2.1.1 should be understood in the following way: if $\widehat{f}$ is a measurable function such that the right hand side of (2.2) is finite, and equality occurs in (2.2), then $\widehat{f}$ must be a Gaussian, and so is f. Here we shall always refer as Gaussians the functions of the form

$$f(x) = e^{A|x|^2 + b\cdot x + C}\,,$$

where $A, C \in \mathbb{C}$, $b \in \mathbb{C}^n$ and $\Re(A) < 0$. The term A is the covariance of the Gaussian f.

Some interesting inequalities arise from Theorem 2.1.1. First, we present the sharp forms of the classical Strichartz inequalities in low dimensions.

Corollary 2.1.2. *In dimension $n = 1$ we have*

$$\|u(t,x)\|_{L^6_t L^6_x(\mathbb{R}\times\mathbb{R})} \le 12^{-1/12} \|f\|_{L^2(\mathbb{R})}, \tag{2.5}$$

and

$$\|u(t,x)\|_{L^8_t L^4_x(\mathbb{R}\times\mathbb{R})} \le 2^{-1/4} \|f\|_{L^2(\mathbb{R})}. \tag{2.6}$$

In dimension $n = 2$ we have

$$\|u(t,x)\|_{L^4_t L^4_x(\mathbb{R}\times\mathbb{R}^2)} \le 2^{-1/2} \|f\|_{L^2(\mathbb{R}^2)}. \tag{2.7}$$

These inequalities are sharp and equality occurs if and only if f is a Gaussian.

The sharp forms (2.5) and (2.7) are the ones discovered by Foschi [26] and Hundertmark-Zharnitsky [38]. They are a direct consequence of Theorem 2.1.1. The novelty here is (2.6), which is obtained by taking $f(x,y) = g(x)g(y)$ in (2.7) and exploiting the product structure of the problem. It is interesting to notice the persistence of the Gaussian maximizers in a case where $q \neq r$.

By using the fact that

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} g(x)\, g(y)\, x \cdot y \, \mathrm{d}x \, \mathrm{d}y \geq 0 \,, \tag{2.8}$$

for any real valued function g, with equality for example if g is radial, one obtains some sharp Sobolev-Strichartz inequalities in low dimensions.

Corollary 2.1.3. *In dimension* $n = 1$ *we have*

$$\|u(t,x)\|_{L_t^{10} L_x^{10}(\mathbb{R}\times\mathbb{R})} \leq (2\sqrt{5}\pi)^{-1/10} \|f'\|_{L^2(\mathbb{R})}^{1/5} \|f\|_{L^2(\mathbb{R})}^{4/5}, \tag{2.9}$$

$$\|u(t,x)\|_{L_t^{12} L_x^{6}(\mathbb{R}\times\mathbb{R})} \leq (6\pi)^{-1/12} \|f'\|_{L^2(\mathbb{R})}^{1/6} \|f\|_{L^2(\mathbb{R})}^{5/6}, \tag{2.10}$$

and

$$\|u(t,x)\|_{L_t^{16} L_x^{4}(\mathbb{R}\times\mathbb{R})} \leq (8\pi)^{-1/16} \|f'\|_{L^2(\mathbb{R})}^{1/8} \|f\|_{L^2(\mathbb{R})}^{7/8}. \tag{2.11}$$

In dimension $n = 2$ *we have*

$$\|u(t,x)\|_{L_t^{6} L_x^{6}(\mathbb{R}\times\mathbb{R}^2)} \leq (12\pi)^{-1/6} \|\nabla f\|_{L^2(\mathbb{R}^2)}^{1/3} \|f\|_{L^2(\mathbb{R}^2)}^{2/3}, \tag{2.12}$$

and

$$\|u(t,x)\|_{L_t^{8} L_x^{4}(\mathbb{R}\times\mathbb{R}^2)} \leq (16\pi)^{-1/8} \|\nabla f\|_{L^2(\mathbb{R}^2)}^{1/4} \|f\|_{L^2(\mathbb{R}^2)}^{3/4}. \tag{2.13}$$

In dimension $n = 4$ *we have*

$$\|u(t,x)\|_{L_t^{4} L_x^{4}(\mathbb{R}\times\mathbb{R}^4)} \leq (32\pi)^{-1/4} \|\nabla f\|_{L^2(\mathbb{R}^4)}^{1/2} \|f\|_{L^2(\mathbb{R}^4)}^{1/2}. \tag{2.14}$$

These inequalities are sharp and equality occurs if and only if f is a Gaussian.

Inequalities (2.9), (2.12) and (2.14) follow directly from Theorem 2.1.1 and relation (2.8). To obtain (2.10) and (2.11) one should put $f(x,y) = g(x)g(y)$ in (2.12) and (2.13), respectively, and exploit the product structure. In an analogous manner one obtains (2.13) by putting $f(x,y,z,k) = g(x,y)g(z,k)$ in (2.14).

2.1.2 Sharp Restriction/Extension Estimates

It has been known for a long time the equivalence of decay inequalities for the space-time norm of the solutions of certain evolution equations and restriction estimates for the Fourier transform over curved surfaces. The classical reference on the subject is Strichartz original paper [61], but seminal ideas can already be observed in the work of Hörmander [36, Corollary 1.3] and C. Fefferman's thesis [25].

The Schrödinger and wave equations are related to the restriction problem for the paraboloid and cone, respectively,

$$S_{parab} := \{(\tau,\omega) \in \mathbb{R}\times\mathbb{R}^n \,:\, \tau = |\omega|^2\}\,, \tag{2.15}$$

and

$$S_{cone} := \{(\tau,\omega) \in \mathbb{R}\times\mathbb{R}^n \,:\, \tau = |\omega|\}\,. \tag{2.16}$$

We endow these surfaces $S \subset \mathbb{R}^{n+1}$ with canonical measures $\mathrm{d}\sigma$ given by

$$\int_{S_{parab}} g(\tau,\omega)\,\mathrm{d}\sigma = \int_{\mathbb{R}^n} g(|\omega|^2,\omega)\,\mathrm{d}\omega\,, \tag{2.17}$$

and

$$\int_{S_{cone}} g(\tau,\omega)\,\mathrm{d}\sigma = \int_{\mathbb{R}^n} g(|\omega|,\omega)\,\frac{\mathrm{d}\omega}{|\omega|}\,. \tag{2.18}$$

In this setting, the restriction estimates are a priori inequalities of the form

$$\|\widehat{h}|_S\|_{L^{p'}(S;\,\mathrm{d}\sigma)} \leq C_{p,q,S}\|h\|_{L^{q'}(\mathbb{R}^{n+1})}. \tag{2.19}$$

The scaling invariance tells us that the global estimate (2.19) can only hold for $p' = nq/(n+2)$ in the case of the paraboloid and $p' = (n-1)q/(n+1)$ in the case of the cone. On the other hand, Knapp's example (see [64]) shows that we must have $q > (2n+2)/n$ for the paraboloid and $q > 2n/(n-1)$ for the cone. The restriction conjecture asserts that these are sufficient conditions in each case for (2.19) to hold, and so far it has been proved for the range $q > (2n+6)/(n+1)$ in both cases, the paraboloid by Tao [65] and the cone by Wolff [69]. We refer the reader to [64] for a survey on the recent progress on the restriction conjecture.

A duality argument using Parseval's identity shows that

$$\begin{aligned} C_{p,q,S} &= \sup_{\|h\|_{L^{q'}(\mathbb{R}^{n+1})}=1} \|\widehat{h}|_S\|_{L^{p'}(S;\,\mathrm{d}\sigma)} \\ &= \sup_{\|h\|_{L^{q'}(\mathbb{R}^{n+1})}=1}\ \sup_{\|g\|_{L^p(S;\,\mathrm{d}\sigma)}=1} \left|\int_S \widehat{h}(\tau,\omega)\,g(\tau,\omega)\,\mathrm{d}\sigma\right| \\ &= \sup_{\|g\|_{L^p(S;\,\mathrm{d}\sigma)}=1}\ \sup_{\|h\|_{L^{q'}(\mathbb{R}^{n+1})}=1} \left|\int_{\mathbb{R}^{n+1}} h(t,x)\,\widehat{g\mathrm{d}\sigma}\,(t,x)\,\mathrm{d}t\,\mathrm{d}x\right| \\ &= \sup_{\|g\|_{L^p(S;\,\mathrm{d}\sigma)}=1} \|\widehat{g\mathrm{d}\sigma}\|_{L^q(\mathbb{R}^{n+1})}\,. \end{aligned} \tag{2.20}$$

Therefore (2.19) is equivalent to the extension estimate

$$\|\widehat{g\mathrm{d}\sigma}\|_{L^q(\mathbb{R}^{n+1})} \leq C_{p,q,S}\|g\|_{L^p(S;\,\mathrm{d}\sigma)}\,, \tag{2.21}$$

for all smooth functions g on S, where $\widehat{g\mathrm{d}\sigma}$ is the Fourier transform of the measure $g\mathrm{d}\sigma$:

$$\widehat{(g\mathrm{d}\sigma)}(t,x) := \frac{1}{(2\pi)^{(n+1)/2}} \int_S g(\tau,\omega)\, e^{-i(t\tau+\omega\cdot x)}\, \mathrm{d}\sigma.$$

In the case of the paraboloid, from (2.4) we see that the solution of the Schrödinger equation (2.1) satisfies

$$u(t,-x) = (2\pi)^{1/2}\widehat{g\mathrm{d}\sigma}(t,x)\,,$$

with $g(|\omega|^2,\omega) = \widehat{f}(\omega)$. Therefore, (2.21) is equivalent to the inequality

$$\|u(t,x)\|_{L^q_t L^q_x(\mathbb{R}\times\mathbb{R}^n)} \leq (2\pi)^{1/2} C_{p,q,S} \|\widehat{f}\|_{L^p(\mathbb{R}^n)}. \tag{2.22}$$

From the equivalence of (2.19), (2.21) and (2.22), the sharp forms (2.5) and (2.7) discovered by Foschi [26] and Hundertmark-Zharnitsky [38] immediately translate into sharp restriction/extension estimates for the paraboloid.

Theorem 2.1.4. *Let S be the paraboloid defined in (2.15) endowed with the measure $\mathrm{d}\sigma$ defined in (2.17). We have*

$$\|\widehat{g\mathrm{d}\sigma}\|_{L^6(\mathbb{R}^2)} \leq (2\pi)^{-1/2} 12^{-1/12} \|g\|_{L^2(S;\,\mathrm{d}\sigma)}\,, \tag{2.23}$$

and

$$\|\widehat{g\mathrm{d}\sigma}\|_{L^4(\mathbb{R}^3)} \leq (4\pi)^{-1/2} \|g\|_{L^2(S;\,\mathrm{d}\sigma)}. \tag{2.24}$$

These inequalities are sharp. Equality occurs in (2.23) and (2.24) if and only if

$$g(|\omega|^2,\omega) = e^{A|\omega|^2+b\cdot\omega+C}\,, \tag{2.25}$$

where $A, C \in \mathbb{C}$, $b \in \mathbb{C}^n$ and $\Re(A) < 0$.

For simplicity, we presented above the sharp extension inequality. One can deduce the dual sharp restriction inequality (2.19) for the paraboloid and find the maximizing functions $h(t,x)$ by using the condition for equality in the duality argument (2.20) (Hölder's inequality)

$$h = C\,|\widehat{g\mathrm{d}\sigma}|^{\frac{q}{q'}-1}\,\overline{\widehat{g\mathrm{d}\sigma}}\,, \tag{2.26}$$

for a complex constant C and g given by (2.25).

In the same spirit, sharp restriction/extension inequalities for the cone are implicit in Foschi's work [26] for the wave equation.

Theorem 2.1.5. *Let S be the cone defined in* (2.16) *endowed with the measure* $\mathrm{d}\sigma$ *defined in* (2.18). *We have*

$$\|\widehat{g\mathrm{d}\sigma}\|_{L^6(\mathbb{R}^3)} \leq (2\pi)^{1/3}\|g\|_{L^2(S;\,\mathrm{d}\sigma)}\,, \tag{2.27}$$

and

$$\|\widehat{g\mathrm{d}\sigma}\|_{L^4(\mathbb{R}^4)} \leq (2\pi)^{1/4}\|g\|_{L^2(S;\,\mathrm{d}\sigma)}. \tag{2.28}$$

These inequalities are sharp. Equality occurs in (2.27) *and* (2.28) *if and only if*

$$g(|\omega|,\omega) = e^{A|\omega|+b\cdot\omega+C}\,, \tag{2.29}$$

where $A, C \in \mathbb{C}$, $b \in \mathbb{C}^n$ and $|\Re(b)| < -\Re(A)$.

We refer the reader to [15] for a brief proof of Theorem 2.1.5, that indicates the basic changes that have to be made in Foschi's argument. Again, the maximizers $h(t,x)$ for the dual restriction inequalities (2.19) can be obtained from the duality condition (2.26) with g given by (2.29).

2.2 Proof of Theorem 2.1.1 - The Sharp Inequality

The proof of Theorem 2.1.1 given here follows closely the outline of Hundertmark and Zharnitsky [38]. As we are interested in an a priori estimate, in this section we suppose that $f \in C_0^\infty(\mathbb{R}^n)$. Throughout the proof of Theorem 2.1.1 we reserve the variables η and ξ to be in $\mathbb{R}^{nk}$ and write $\eta = (\eta_1, \eta_2, ..., \eta_k)$ with each $\eta_i \in \mathbb{R}^n$. We have also defined $F(\eta) = f(\eta_1)f(\eta_2)...f(\eta_k)$ and $K(\eta) = \frac{1}{k}\sum_{1\le i<j\le k} |\eta_i - \eta_j|^2$. Let us write

$$F_1(\eta) = \widehat{F}(\eta)K(\eta)^{\frac{n(k-1)-2}{4}}.$$

In the space $L^2(\mathbb{R}^{nk})$, let E be the closed subspace consisting of the functions invariant under any orthonormal transformation (rotation here for short) R that fixes the vectors $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}^{nk}$ given by

$$\alpha_i = (e_i, e_i, ..., e_i) \quad (k \ \text{ times}),$$

where $e_i = (0, 0, ..., 1, ..., 0)$ is the i-th canonical vector in $\mathbb{R}^n$. In other words,

$$E = \text{closure}\left(\text{span}\{H(\eta\cdot\alpha_1, ..., \eta\cdot\alpha_n, |\eta|^2);\ H \in C_0^\infty(\mathbb{R}\times ...\times\mathbb{R}\times\mathbb{R}^+)\}\right).$$

Denote by $P_E : L^2(\mathbb{R}^{nk}) \to L^2(\mathbb{R}^{nk})$ the orthogonal projection operator onto the subspace E. The heart of the matter is the following representation lemma.

Lemma 2.2.1 (Representation Lemma)**.** *Let $u : \mathbb{R}\times\mathbb{R}^n \to \mathbb{C}$ be the solution of the Schrödinger equation* (2.1). Then

$$\int_{\mathbb{R}\times\mathbb{R}^n} |u(t,x)|^{2k}\, \mathrm{d}x\, \mathrm{d}t = C_{n,k}\, \langle P_E(F_1), F_1\rangle_{L^2(\mathbb{R}^{nk})}.$$

with the constant $C_{n,k}$ defined in (2.3).

Proof. Using the representation (2.4) for the solution $u(t,x)$ we obtain

$$|u(t,x)|^{2k} = \frac{1}{(2\pi)^{nk}} \int_{\mathbb{R}^{nk}\times\mathbb{R}^{nk}} e^{ix\cdot(\sum \eta_i - \sum \xi_i)}\, e^{-it(|\eta|^2-|\xi|^2)}\, \widehat{F}(\eta)\overline{\widehat{F}(\xi)}\,\mathrm{d}\eta\,\mathrm{d}\xi,$$

where $\eta = (\eta_1, \eta_2, ..., \eta_k)$ and $\xi = (\xi_1, \xi_2, ..., \xi_k)$, with each η_i and ξ_i in $\mathbb{R}^n$. Integrating with respect to x and t and using that, as distributions, the n-dimensional delta function $\delta_n(w) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-ix\cdot w}\mathrm{d}x$, one arrives at

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}^n} |u(t,x)|^{2k}\,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{1}{(2\pi)^{n(k-1)-1}} \int_{\mathbb{R}^{nk}\times\mathbb{R}^{nk}} \delta_n\left(\sum_{i=1}^{k}\eta_i - \sum_{i=1}^{k}\xi_i\right) \delta\big(|\eta|^2-|\xi|^2\big)\, \widehat{F}(\eta)\overline{\widehat{F}(\xi)}\,\mathrm{d}\eta\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^{n(k-1)-1}} \int_{\mathbb{R}^{nk}\times\mathbb{R}^{nk}} \left(\prod_{i=1}^{n}\delta\big((\eta-\xi)\cdot\alpha_i\big)\right) \delta\big(|\eta|^2-|\xi|^2\big)\, \widehat{F}(\eta)\overline{\widehat{F}(\xi)}\,\mathrm{d}\eta\,\mathrm{d}\xi.
\end{aligned}$$

We will rewrite the last equation in the following strategic way

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}^n} |u(t,x)|^{2k}\,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{1}{(2\pi)^{n(k-1)-1}} \int_{\mathbb{R}^{nk}\times\mathbb{R}^{nk}} \frac{\big(\prod_{i=1}^{n}\delta\big((\eta-\xi)\cdot\alpha_i\big)\big)\ \delta\big(|\eta|^2-|\xi|^2\big)}{\big(K(\eta)K(\xi)\big)^{\frac{n(k-1)-2}{4}}}\, F_1(\eta)\overline{F_1(\xi)}\,\mathrm{d}\eta\,\mathrm{d}\xi.
\end{aligned}$$

The insight now is to recognize the last expression as a quadratic form associated to a self-adjoint operator. Indeed, for $G \in C_0^\infty(\mathbb{R}^{nk})$ define the operator

$$AG(\xi) = \frac{1}{(2\pi)^{n(k-1)-1}} \int_{\mathbb{R}^{nk}} \frac{\big(\prod_{i=1}^{n}\delta\big((\eta-\xi)\cdot\alpha_i\big)\big)\ \delta\big(|\eta|^2-|\xi|^2\big)}{\big(K(\eta)K(\xi)\big)^{\frac{n(k-1)-2}{4}}}\, G(\eta)\,\mathrm{d}\eta\,. \tag{2.30}$$

In this context we have

$$\int_{\mathbb{R}\times\mathbb{R}^n} |u(t,x)|^{2k}\,\mathrm{d}x\,\mathrm{d}t = \langle AF_1, F_1\rangle_{L^2(\mathbb{R}^{nk})}\,.$$

Our objective is to show that the operator A is a multiple of the projection operator P_E. We start by showing that A is a bounded operator in $L^2(\mathbb{R}^{nk})$, via the following lemma.

Lemma 2.2.2. *(i) For all $\xi \in \mathbb{R}^{nk}$ the measure*

$$m_\xi(\mathrm{d}\eta) = \frac{k^{n/2}\,\Gamma\left(\frac{n(k-1)}{2}\right)}{\pi^{n(k-1)/2}} \frac{\left(\prod_{i=1}^n \delta\big((\eta-\xi)\cdot\alpha_i\big)\right)\,\delta\big(|\eta|^2-|\xi|^2\big)}{\big(K(\eta)K(\xi)\big)^{\frac{n(k-1)-2}{4}}}\,\mathrm{d}\eta$$

is a probability measure on $\mathbb{R}^{nk}$.

(ii) For all Borel measurable sets $B \subset \mathbb{R}^{nk}$, we have

$$\int_{\mathbb{R}^{nk}} m_\xi(B)\,\mathrm{d}\xi = |B|\,,$$

where $|B|$ denotes the Lebesgue measure of B.

Proof. Throughout this proof let us write

$$C = \frac{k^{n/2}\,\Gamma\left(\frac{n(k-1)}{2}\right)}{\pi^{n(k-1)/2}}\,.$$

Observe that in the support of the delta functions we have $\sum \eta_i = \sum \xi_i$ and $|\eta|^2 = |\xi|^2$. This implies that $K(\eta) = K(\xi)$, since

$$\begin{aligned} K(\eta) &= \frac{1}{k}\sum_{1\le i<j\le k} |\eta_i-\eta_j|^2 \\ &= |\eta|^2 - \frac{|\eta_1+\eta_2+...+\eta_k|^2}{k} = |\eta|^2 - \frac{\sum_{i=1}^n(\eta\cdot\alpha_i)^2}{k}\,. \end{aligned} \tag{2.31}$$

Therefore we have

$$m_\xi(\mathbb{R}^{nk}) = \frac{C}{K(\xi)^{\frac{n(k-1)-2}{2}}}\int_{\mathbb{R}^{nk}} \left(\prod_{i=1}^n \delta\big((\eta-\xi)\cdot\alpha_i\big)\right)\,\delta\big(|\eta|^2-|\xi|^2\big)\,\mathrm{d}\eta\,. \tag{2.32}$$

Let $\{\tilde{e}_j\}$, $1 \le j \le nk$, be the canonical vectors in $\mathbb{R}^{nk}$. Change the variable η in the integration (2.32) by a rotation R that sends α_i to $\sqrt{k}\tilde{e}_i$ for $1 \le i \le n$. We obtain

$$
\begin{aligned}
m_\xi(\mathbb{R}^{nk}) &= \frac{C}{K(\xi)^{\frac{n(k-1)-2}{2}}} \int_{\mathbb{R}^{nk}} \delta_n \left(\sqrt{k}\eta_1 - \sum \xi_i\right) \delta\big(|\eta|^2 - |\xi|^2\big)\, \mathrm{d}\eta \\
&= \frac{C}{k^{n/2} K(\xi)^{\frac{n(k-1)-2}{2}}} \int_{\mathbb{R}^{n(k-1)}} \delta \left(\sum_{i=2}^{k} |\eta_i|^2 - K(\xi)\right) \mathrm{d}\eta_2\, \mathrm{d}\eta_3 ... \mathrm{d}\eta_k \\
&= \frac{C\left|S^{n(k-1)-1}\right|}{k^{n/2} K(\xi)^{\frac{n(k-1)-2}{2}}} \int_0^\infty \delta(r^2 - K(\xi))\, r^{n(k-1)-1} \mathrm{d}r \\
&= \frac{C\left|S^{n(k-1)-1}\right|}{2\, k^{n/2} K(\xi)^{\frac{n(k-1)-2}{2}}} \int_0^\infty \delta(t - K(\xi))\, t^{\frac{n(k-1)-2}{2}} \mathrm{d}t \\
&= \frac{C\left|S^{n(k-1)-1}\right|}{2\, k^{n/2}} = 1\,,
\end{aligned}
$$

and this proves (i). To prove (ii), just observe the symmetry of the measure m with respect to the variables η and ξ,

$$
\begin{aligned}
\int_{\mathbb{R}^{nk}} m_\xi(B)\, \mathrm{d}\xi &= \int_{\mathbb{R}^{nk}} \int_B \frac{C\, \left(\prod_{i=1}^n \delta\big((\eta - \xi)\cdot \alpha_i\big)\right) \delta\big(|\eta|^2 - |\xi|^2\big)}{\big(K(\eta)K(\xi)\big)^{\frac{n(k-1)-2}{4}}}\, \mathrm{d}\eta\, \mathrm{d}\xi \\
&= \int_B \int_{\mathbb{R}^{nk}} \frac{C\, \left(\prod_{i=1}^n \delta\big((\eta - \xi)\cdot \alpha_i\big)\right) \delta\big(|\eta|^2 - |\xi|^2\big)}{\big(K(\eta)K(\xi)\big)^{\frac{n(k-1)-2}{4}}}\, \mathrm{d}\xi\, \mathrm{d}\eta \\
&= \int_B m_\eta(\mathbb{R}^{nk})\, \mathrm{d}\eta = \int_B \mathrm{d}\eta = |B|.
\end{aligned}
$$

$\square$

We now return to the proof of the Representation Lemma 2.2.1. Note the the operator A can be written as

$$AG(\xi) = C_{n,k} \int_{\mathbb{R}^{nk}} G(\eta) m_\xi(\mathrm{d}\eta)\,.$$

The boundedness of the operator A in $L^2(\mathbb{R}^{nk})$ follows from an application of Lemma 2.2.2 and Jensen's inequality

$$\begin{aligned}
\|AG\|^2_{L^2(\mathbb{R}^{nk})} &= C^2_{n,k} \int_{\mathbb{R}^{nk}} \left| \int_{\mathbb{R}^{nk}} G(\eta) m_\xi(\mathrm{d}\eta) \right|^2 \mathrm{d}\xi \\
&\le C^2_{n,k} \int_{\mathbb{R}^{nk}} \int_{\mathbb{R}^{nk}} |G(\eta)|^2 m_\xi(\mathrm{d}\eta)\, \mathrm{d}\xi \\
&= C^2_{n,k} \int_{\mathbb{R}^{nk}} |G(\eta)|^2 \int_{\mathbb{R}^{nk}} m_\xi(\mathrm{d}\eta)\, \mathrm{d}\xi \\
&= C^2_{n,k} \int_{\mathbb{R}^{nk}} |G(\eta)|^2 \mathrm{d}\eta = C^2_{n,k} \|G\|^2_{L^2(\mathbb{R}^{nk})}.
\end{aligned}$$

We thus arrive at

$$\|AG\|_{L^2(\mathbb{R}^{nk})} \le C_{n,k} \|G\|_{L^2(\mathbb{R}^{nk})},$$

proving that the operator A extends to a bounded operator from $L^2(\mathbb{R}^{nk})$ to $L^2(\mathbb{R}^{nk})$. It remains to show that A is a multiple of the projection operator P_E. Let R be a rotation on $\mathbb{R}^{nk}$ fixing the vectors $\alpha_1, ..., \alpha_n$. It is clear from (2.30) and (2.31) that

$$AG(R\xi) = AG(\xi),$$

therefore A maps $L^2(\mathbb{R}^{nk})$ into the subspace E. From the fact that the operator A is self-adjoint we can show that $A(E^\perp) = 0$. It remains to prove that

A acts like a multiple of the identity on E. For this, consider a function $H \in C_0^\infty(\mathbb{R} \times \mathbb{R} \times ... \times \mathbb{R} \times \mathbb{R}^+)$ and write

$$G(\eta) = H(\eta \cdot \alpha_1, \eta \cdot \alpha_2, ..., \eta \cdot \alpha_n, |\eta|^2). \tag{2.33}$$

From definition (2.30) we find that, for a G of the form (2.33),

$$AG(\xi) = C_{n,k} G(\xi).$$

Since the functions of the form (2.33) are dense in E, we conclude that $A = C_{n,k} I$ on E. We have proved that $A = C_{n,k} P_E$ and this concludes the lemma.

□

The proof of the inequality proposed in Theorem 2.1.1 is then a trivial consequence of the Representation Lemma 2.2.1. In fact,

$$\begin{aligned} \int_{\mathbb{R}\times\mathbb{R}^n} |u(t,x)|^{2k} \, \mathrm{d}x \, \mathrm{d}t &= C_{n,k} \langle P_E(F_1), F_1 \rangle_{L^2(\mathbb{R}^{nk})} \leq C_{n,k} \|F_1\|^2_{L^2(\mathbb{R}^{nk})} \\ &= C_{n,k} \int_{\mathbb{R}^{nk}} |\widehat{F}(\eta)|^2 \, K(\eta)^{\frac{n(k-1)-2}{2}} \, \mathrm{d}\eta \, . \end{aligned} \tag{2.34}$$

It remains to investigate when equality in (2.34) can be attained. A necessary and sufficient condition is that the function $F_1(x)$ belongs to the subspace E.

2.3 Proof of Theorem 2.1.1 - Gaussian Maximizers

We investigate here under which conditions the function

$$F_1(\eta) = \widehat{F}(\eta) K(\eta)^{\frac{n(k-1)-2}{4}}$$

belongs to the subspace E. Let us say that a measurable function $G : \mathbb{R}^{nk} \to \mathbb{C}$ satisfies the property $(\star)$ if G is invariant under all the rotations R that fix the vectors $\alpha_1, \alpha_2, ..., \alpha_n$. In this setting, $G \in E$ if and only if $G \in L^2(\mathbb{R}^{nk})$ and satisfies $(\star)$.

From (2.31) we see that $K(x)$ satisfies $(\star)$. Therefore, we must have $\widehat{F}(\eta) = \widehat{f}(\eta_1)\widehat{f}(\eta_2)...\widehat{f}(\eta_k)$ satisfying $(\star)$, and we shall prove that under these symmetries $\widehat{f}$ must be a Gaussian . The proof will be divided in five steps.

Step 1. Let $g : \mathbb{R}^n \to \mathbb{C}$ be a measurable function such that $G(\eta) = g(\eta_1)g(\eta_2)...g(\eta_k)$ satisfies

$$\int_{\mathbb{R}^{nk}} |G(\eta)|^2 \, K(\eta)^{\frac{n(k-1)-2}{2}} \, \mathrm{d}\eta < \infty \, . \tag{2.35}$$

Then $g \in L^p(\mathbb{R}^n)$ for $p = \frac{2nk}{2nk-n-2}$.

This is a consequence of the following three inequalities:

(i) A basic inequality for real numbers:

$$K(\eta)^{\frac{n(k-1)-2}{2}} \geq C \sum_{1 \leq i < j \leq n} |\eta_i - \eta_j|^{n(k-1)-2} \, ;$$

(ii) The reversed Hardy-Littlewood-Sobolev inequality of W. Beckner (personal communication):

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |g(x)||x-y|^{\lambda}|h(y)| \, \mathrm{d}x \, \mathrm{d}y \geq C(n,\lambda) \, \|g\|_{L^{\frac{2n}{2n+\lambda}}(\mathbb{R}^n)} \|h\|_{L^{\frac{2n}{2n+\lambda}}(\mathbb{R}^n)} \, ,$$

where $\lambda > 0$, the sharp constant given by

$$C(n,\lambda) = \pi^{\lambda/2} \frac{\Gamma(n/2+\lambda/2)}{\Gamma(n+\lambda/2)} \left[\frac{\Gamma(n)}{\Gamma(n/2)}\right]^{1+\lambda/n} \, ,$$

and the only maximizers being $g(x) = c\,h(x)$, $c \in \mathbb{C}$ a constant, and

$$h(x) = A(B^2 + |x - x_0|^2)^{-(2n+\lambda)/2}\,,$$

for some $A \in \mathbb{C}$, $0 \neq B \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$. For our purposes it suffices to use this inequality in the following format

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |\widehat{f}(\eta_i)|^2 |\widehat{f}(\eta_j)|^2 |\eta_i - \eta_j|^{n(k-1)-2}\, \mathrm{d}\eta_i\, \mathrm{d}\eta_j \geq C \|\widehat{f}\|^4_{L^r(\mathbb{R}^n)}\,,$$

where $r = 4n/(n(k+1) - 2)$;

(iii) Hölder's inequality:

$$\|\widehat{f}\|^{2k}_{L^p(\mathbb{R}^n)} \leq \|\widehat{f}\|^4_{L^r(\mathbb{R}^n)} \|\widehat{f}\|^{2k-4}_{L^2(\mathbb{R}^n)}.$$

A simple combination of (i), (ii) and (iii) above provides (2.35). From now on we fix $p = \frac{2nk}{2nk-n-2}$.

Step 2. Let $g \in L^p(\mathbb{R}^n)$ be such that $G(\eta)$ satisfies the property $(\star)$. Then g is a product of one-dimensional functions.

We shall write here each $\eta_i \in \mathbb{R}^n$ as $\eta_i = (\eta_{i1}, \eta_{i2}, ..., \eta_{in})$. If $g \in L^p(\mathbb{R}^n)$ is nonzero, there exists a cube $J = \prod_{i=1}^n [a_i, b_i] \subset \mathbb{R}^n$ such that

$$\int_J g(y)\, \mathrm{d}y = A \neq 0.$$

Consider the orthonormal transformation R in $\mathbb{R}^{nk}$ that simply switches the coordinates η_{11} and η_{21} on $\eta = (\eta_1, ..., \eta_k)$. Naturally, this transformation fixes the vectors α_i and thus the relation $G(Rx) = G(x)$ implies

$$\begin{aligned} &g(\eta_{11}, \eta_{12}, ..., \eta_{1n})g(\eta_{21}, \eta_{22}, ..., \eta_{2n})g(\eta_3)...g(\eta_k) \\ &\qquad = g(\eta_{21}, \eta_{12}, ..., \eta_{1n})g(\eta_{11}, \eta_{22}, ..., \eta_{2n})g(\eta_3)...g(\eta_k). \end{aligned} \tag{2.36}$$

Integrating both sides of (2.36) with respect to $\mathrm{d}\eta_2\mathrm{d}\eta_3...\mathrm{d}\eta_k$ on $J \times J \times ... \times J$ we find that

$$\begin{aligned} &A^{k-1}g(\eta_{11}, \eta_{12}, ..., \eta_{1n}) \\ &\qquad = A^{k-2}\int_{a_1}^{b_1} g(\eta_{21}, \eta_{12}, ..., \eta_{1n})\,\mathrm{d}\eta_{21}\int_{J'} g(\eta_{11}, \eta_{22}, ..., \eta_{2n})\,\mathrm{d}\eta_2'\,, \end{aligned} \tag{2.37}$$

where $J' = \prod_{i=2}^n [a_i, b_i]$ and $\mathrm{d}\eta_2' = \mathrm{d}\eta_{22}\mathrm{d}\eta_{23}...\mathrm{d}\eta_{2n}$. Expression (2.37) plainly says that

$$g(\eta_{11}, \eta_{12}, ..., \eta_{1n}) = w_1(\eta_{11})\,h_1(\eta_{12}, ..., \eta_{1n}). \tag{2.38}$$

By repeating this argument we arrive at

$$g(\eta_{11}, \eta_{12}, ..., \eta_{1n}) = w_j(\eta_{1j})\,h_j(\eta_{11}, ..., \eta_{1(j-1)}, \eta_{1(j+1)}, ..., \eta_{1n}), \tag{2.39}$$

for $j = 2, ..., n$. Expressions (2.38) and (2.39) are sufficient to conclude that

$$g(\eta_{11}, \eta_{12}, ..., \eta_{1n}) = g_1(\eta_{11})g_2(\eta_{12})...g_n(\eta_{1n}).$$

Step 3. Suppose that all g_i's are smooth and non-vanishing. Then all g_i's are Gaussians with the same covariance. Therefore g is itself a Gaussian .

Let R_{12} be a rotation on $\mathbb{R}^{2n}$ fixing the vectors $\beta_i = \frac{1}{\sqrt{2}}(e_i, e_i)$, $i =$

$1, 2, ..., n$. Observe that the rotation on $\mathbb{R}^{nk}$ given by

$$R = \begin{bmatrix} R_{12} & & & & \\ & I & & 0 & \\ & & I & & \\ & 0 & & \ddots & \\ & & & & I \end{bmatrix} \tag{2.40}$$

fixes the vectors $\alpha_i = (e_i, e_i, ..., e_i) \in \mathbb{R}^{nk}$. Among all the possible rotations R given by this form, we will choose a simple rotation R_{12} to work with. Let us denote the tensor product $a \otimes b$ of two vectors $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ in $\mathbb{R}^n$ as the $n \times n$ matrix $[a_i b_j]$, corresponding to the linear transformation $x \mapsto \langle x, b\rangle a$. Consider the orthonormal basis of $\mathbb{R}^{2n}$ formed by the vectors $\beta_i = \frac{1}{\sqrt{2}}(e_i, e_i)$ and $\gamma_i = \frac{1}{\sqrt{2}}(e_i, -e_i)$, with $i = 1, 2, ..., n$, and let $R_{12}(\theta)$ be given by

$$\begin{aligned} R_{12}(\theta) = &\sum_{i=1}^{n} \beta_i \otimes \beta_i + \sum_{i=3}^{n} \gamma_i \otimes \gamma_i \\ &+ \cos(\theta)\gamma_1 \otimes \gamma_1 - \sin(\theta)\gamma_1 \otimes \gamma_2 + \sin(\theta)\gamma_2 \otimes \gamma_1 + \cos(\theta)\gamma_2 \otimes \gamma_2. \end{aligned}$$

Let $R(\theta)$ be the rotation on $\mathbb{R}^{nk}$ given by the matrix (2.40) with the corresponding $R_{12}(\theta)$. From the fact that $G(R(\theta)\eta) = G(\eta)$ and $R(0) = I$ we obtain

$$\begin{aligned} 0 = &-2\frac{\partial G(R(\theta)\eta)}{\partial\theta}|_{\theta=0} \\ &= \left[(\eta_{12} - \eta_{22})\partial_{\eta_{11}} - (\eta_{11} - \eta_{21})\partial_{\eta_{12}} - (\eta_{12} - \eta_{22})\partial_{\eta_{21}} + (\eta_{11} - \eta_{21})\partial_{\eta_{22}}\right] G(\eta). \end{aligned}$$

By introducing the logarithmic derivatives $h_i' = g_i'/g_i$ the last expression becomes

$$(\eta_{12}-\eta_{22})h_1'(\eta_{11})-(\eta_{11}-\eta_{21})h_2'(\eta_{12})-(\eta_{12}-\eta_{22})h_1'(\eta_{21})+(\eta_{11}-\eta_{21})h_2'(\eta_{22}) = 0.$$

Differentiating with respect to the variable η_{11} we obtain

$$(\eta_{12} - \eta_{22})h_1''(\eta_{11}) - h_2'(\eta_{12}) + h_2'(\eta_{22}) = 0.$$

Finally, differentiating with respect to η_{22} yields

$$h_1''(\eta_{11}) = h_2''(\eta_{22})\,,$$

and since the variables η_{11} and η_{22} are independent we conclude that both logarithmic second derivatives are constant. The argument above can be reproduced for γ_1 and γ_j yielding $h_1'' = h_j'' = C$ for all $j = 1, 2, ..., n$. This proves that all g_i's are Gaussians with the same covariance, and thus g will itself be a Gaussian.

The last two steps (reduction to the smooth non-vanishing case) plainly follow the argument of Hundertmark and Zharnitsky [38]. This idea originally appeared in a paper by Carlen [13]. We denote by P_ϵ the convolution with the Gaussian kernel on $\mathbb{R}^{nk}$

$$\varphi_\epsilon(\eta) = \frac{1}{(2\pi\epsilon)^{nk/2}}\, e^{-\frac{|\eta|^2}{2\epsilon}}\,,$$

and by Q_ϵ the convolution with the Gaussian kernel on $\mathbb{R}^n$

$$\phi_\epsilon(y) = \frac{1}{(2\pi\epsilon)^{n/2}}\, e^{-\frac{|y|^2}{2\epsilon}}\,.$$

Step 4. Let $g \in L^p(\mathbb{R}^n)$ be such that $G(\eta)$ satisfies the property $(\star)$. Assume $Q_\epsilon(g)$ never vanishes as $\epsilon \to 0$. Then g is a Gaussian.

Observe that $P_\epsilon(G)$ inherits the rotational symmetries of G, and since

$$P_\epsilon(G)(\eta) = Q_\epsilon(g)(\eta_1)\, Q_\epsilon(g)(\eta_2) \ldots Q_\epsilon(g)(\eta_k)\,, \tag{2.41}$$

and $Q_\epsilon(g)$ is smooth and non-vanishing, we conclude by Step 3 that it must be a Gaussian. As $g \in L^p(\mathbb{R}^n)$, we have $g = \lim_{\epsilon\to 0} Q_\epsilon(g)$ and this implies that g, being a limit of Gaussians, is also a Gaussian.

Step 5. Let $g \in L^p(\mathbb{R}^n)$ be such that $G(\eta)$ satisfies the property $(\star)$. Then $Q_\epsilon(g)$ never vanishes as $\epsilon \to 0$.

Indeed, take absolute values in (2.41) and apply the convolution operator P_λ again

$$P_\lambda|P_\epsilon(G)|(\eta) = Q_\lambda|Q_\epsilon(g)|(\eta_1)\, Q_\lambda|Q_\epsilon(g)|(\eta_2) \ldots Q_\lambda|Q_\epsilon(g)|(\eta_k)\,.$$

Again, $P_\lambda|P_\epsilon(G)|$ inherits all the rotational symmetries of $P_\epsilon(G)$, in particular those of G. Since $Q_\epsilon(g) \to g$ in $L^p(\mathbb{R}^n)$, as $\epsilon \to 0$, we conclude that $Q_\epsilon(g)$ is not the zero function for small ϵ. Since convolution with a Gaussian improves positivity, $Q_\lambda|Q_\epsilon(g)|$ is a strictly positive smooth function. By Step 4 we conclude that $|Q_\epsilon(g)|$ is a Gaussian, and thus never vanishes for small ϵ.

By putting $g = \widehat{f}$ in Steps 1-5 we are led to the conclusion that $\widehat{f}$ must be a Gaussian, and then so is f.

Chapter 3

Entire Approximations of Exponential Type

3.1 Preliminaries

3.1.1 History of the Problem

An entire function $F : \mathbb{C} \to \mathbb{C}$ is said to be of *exponential type* $2\pi\delta \geq 0$ if, for any $\epsilon > 0$, there exists a constant C_ϵ such that

$$|F(z)| \leq C_\epsilon e^{(2\pi\delta+\epsilon)|z|}$$

for all $z \in \mathbb{C}$. Given a function $f : \mathbb{R} \to \mathbb{R}$ we address here the problem of finding an entire function $F(z)$ of exponential type at most $2\pi\delta$ such that the integral

$$\int_{-\infty}^{\infty} |F(x) - f(x)|\,\mathrm{d}x \tag{3.1}$$

is minimized. A typical variant of this problem occurs when we impose the additional condition that $F(z)$ is real on $\mathbb{R}$ and satisfies $F(x) \geq f(x)$ for all $x \in \mathbb{R}$. In this case a minimizer of the integral (3.1) is called an extremal majorant of exponential type for $f(x)$. Extremal minorants are defined analogously.

In the special case $f(x) = \mathrm{sgn}(x)$, an explicit solution to this problem was found in the 1930's by A. Beurling, but his results were not published at the time of their discovery. Later, Beurling's solution was rediscovered by

A. Selberg, who recognized its importance in connection with the large sieve inequality of analytic number theory. In particular, Selberg observed that Beurling's function could be used to majorize and minorize the function

$$\tfrac{1}{2}\mathrm{sgn}(x-a)+\tfrac{1}{2}\mathrm{sgn}(b-x)=\begin{cases}1 & \text{if } a<x<b,\\ \frac{1}{2} & \text{if } x=a \text{ or } x=b,\\ 0 & \text{if } x<a \text{ or } b<x,\end{cases} \tag{3.2}$$

where $a<b$. Of course, this function is essentially the characteristic function of the interval with endpoints a and b. The functions that majorize and minorize (3.2) are real entire functions of exponential type at most $2\pi\delta$, but in applications it is often useful to exploit the fact that their Fourier transforms are continuous functions supported on the interval $[-\delta,\delta]$ (Paley-Wiener theorem). An account of these functions, the history of their discovery, and many other applications can be found in the classical paper [67] by J. D. Vaaler.

Since the Beurling-Selberg breakthrough, the theory of extremal approximations has been extended to different classes of functions, providing new applications in analysis, number theory and equidistribution theory. The problem for the function $f(x)=e^{-\lambda|x|}$, $\lambda>0$, was discussed by Graham and Vaaler in [30] and the author and Vaaler in [19]. The problem for $f(x)=x^n\,\mathrm{sgn}(x)$, where $n\in\mathbb{N}$, was considered by F. Littmann in [48], [49] and [50]. The construction of the extremal approximations for a class of even functions that includes $f(x)=\log|x|$ and $f(x)=|x|^{\alpha}$, with $-1<\alpha<1$, was achieved by the author and Vaaler in [18] and [19]. Other problems on approximation by entire functions and trigonometric polynomials have been discussed in [14], [29], [31],

[44], [53], [58] and [68]. Extensions of this problem to several variables were considered in [6], [35] and [45].

3.1.2 The Extremal Problem for the Gaussian

Throughout this chapter we will consider the Fourier transform of a function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$\widehat{f}(t) = \int_{-\infty}^{\infty} f(x)\, e(-xt)\, \mathrm{d}x\,,$$

where $e(z) = e^{2\pi i z}$. We consider here the problem of majorizing, minorizing, and L^1-approximating the Gaussian function

$$x \mapsto G_\lambda(x) = e^{-\pi\lambda x^2}$$

on $\mathbb{R}$ by entire functions of exponential type. Here $\lambda > 0$ is a fixed parameter. For each positive value of λ we define three entire functions as follows:

$$K_\lambda(z) = \Big(\frac{\cos \pi z}{\pi}\Big)\Bigg\{\sum_{n=-\infty}^{\infty} (-1)^{n+1} \frac{G_\lambda\big(n+\frac{1}{2}\big)}{\big(z-n-\frac{1}{2}\big)}\Bigg\}, \tag{3.3}$$

$$L_\lambda(z) = \Big(\frac{\cos \pi z}{\pi}\Big)^2\Bigg\{\sum_{m=-\infty}^{\infty} \frac{G_\lambda\big(m+\frac{1}{2}\big)}{\big(z-m-\frac{1}{2}\big)^2} + \sum_{n=-\infty}^{\infty} \frac{G_\lambda'\big(n+\frac{1}{2}\big)}{\big(z-n-\frac{1}{2}\big)}\Bigg\}, \tag{3.4}$$

$$M_\lambda(z) = \Big(\frac{\sin \pi z}{\pi}\Big)^2\Bigg\{\sum_{m=-\infty}^{\infty} \frac{G_\lambda(m)}{(z-m)^2} + \sum_{n=-\infty}^{\infty} \frac{G_\lambda'(n)}{(z-n)}\Bigg\}. \tag{3.5}$$

The function $K_\lambda(z)$ is an entire function of exponential type π which interpolates the values of the function $G_\lambda(z)$ on the set of shifted integers $\mathbb{Z} + \frac{1}{2}$. We will show that among all entire functions of exponential type at most π, the function $K_\lambda(z)$ provides the best approximation to $G_\lambda(z)$ with respect to the L^1-norm on $\mathbb{R}$.

The function $L_\lambda(z)$ is a real entire function of exponential type 2π which interpolates both the values of $G_\lambda(z)$ and the values of its derivative $G'_\lambda(z)$ on the set of shifted integers $\mathbb{Z}+\frac{1}{2}$. Similarly, the function $M_\lambda(z)$ is a real entire function of exponential type 2π which interpolates both the values of $G_\lambda(z)$ and the values of its derivative $G'_\lambda(z)$ on the set of integers $\mathbb{Z}$. By a *real* entire function we understand an entire function whose restriction to $\mathbb{R}$ is real valued. We will show that these functions satisfy the basic inequality

$$L_\lambda(x) \le G_\lambda(x) \le M_\lambda(x) \tag{3.6}$$

for all real x. Moreover, we will show that the value of each of the two integrals

$$\int_{-\infty}^{\infty} \Big\{G_\lambda(x) - L_\lambda(x)\Big\}\,\mathrm{d}x \quad \text{and} \quad \int_{-\infty}^{\infty} \Big\{M_\lambda(x) - G_\lambda(x)\Big\}\,\mathrm{d}x,$$

is minimized.

In order to state a more precise form of our main results we make use of the basic theta functions. Here v is a complex variable, τ is a complex variable with $\Im\{\tau\} > 0$ and $q = e^{\pi i \tau}$. Our notation for the theta functions follows that of Chandrasekharan [21]. Thus we define

$$\theta_1(v,\tau) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} e\big((n+\tfrac{1}{2})v\big), \tag{3.7}$$

$$\theta_2(v,\tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e(nv), \tag{3.8}$$

$$\theta_3(v,\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} e(nv). \tag{3.9}$$

We note that for a fixed value of τ with $\Im\{\tau\} > 0$, each of the functions $v \mapsto \theta_1(v,\tau)$, $v \mapsto \theta_2(v,\tau)$, and $v \mapsto \theta_3(v,\tau)$ is an *even* entire function of v. The function $v \mapsto \theta_1(v,\tau)$ is periodic with period 2 and satisfies the identity

$$\theta_1(v+1,\tau) = -\theta_1(v,\tau) \tag{3.10}$$

for all complex v. Both of the functions $v \mapsto \theta_2(v,\tau)$, and $v \mapsto \theta_3(v,\tau)$ are periodic with period 1. They are related by the identity

$$\theta_2(v+\tfrac{1}{2},\tau) = \theta_3(v,\tau). \tag{3.11}$$

The transformation formulas for the theta functions [21, Chapter V, Theorem 9, Corollary 1] provide a connection with the function $G_\lambda(z)$. In particular we have

$$\sum_{n=-\infty}^{\infty} (-1)^n G_\lambda(n-v) = \lambda^{-\frac{1}{2}}\theta_1\big(v, i\lambda^{-1}\big), \tag{3.12}$$

$$\sum_{n=-\infty}^{\infty} G_\lambda(n+\tfrac{1}{2}-v) = \lambda^{-\frac{1}{2}}\theta_2\big(v, i\lambda^{-1}\big), \tag{3.13}$$

$$\sum_{n=-\infty}^{\infty} G_\lambda(n-v) = \lambda^{-\frac{1}{2}}\theta_3\big(v, i\lambda^{-1}\big). \tag{3.14}$$

Our first main result identifies the entire function $K_\lambda(z)$ as the unique best approximation to $G_\lambda(z)$ among all entire functions of exponential type at most π.

Theorem 3.1.1. *Let $F(z)$ be an entire function of exponential type at most π. Then*

$$\lambda^{-\frac{1}{2}}\int_{\frac{1}{2}}^{\frac{1}{2}} \theta_1\big(u, i\lambda^{-1}\big)\,\mathrm{d}u \le \int_{-\infty}^{\infty} \big|G_\lambda(x) - F(x)\big|\,\mathrm{d}x, \tag{3.15}$$

and there is equality in (3.15) *if and only if $F(z) = K_\lambda(z)$.*

Next we consider the problem of minorizing and majorizing $G_\lambda(z)$ along the real line by a real entire function of exponential type at most 2π.

Theorem 3.1.2. *(i) Let $F(z)$ be a real entire function of exponential type at most 2π such that*

$$F(x) \leq G_\lambda(x)$$

for all real x. Then

$$\int_{-\infty}^{\infty} F(x)\ \mathrm{d}x \leq \lambda^{-\frac{1}{2}}\theta_2\big(0, i\lambda^{-1}\big), \tag{3.16}$$

and there is equality in (3.16) *if and only if $F(z) = L_\lambda(z)$.*

(ii) Let $F(z)$ be a real entire function of exponential type at most 2π such that

$$G_\lambda(x) \leq F(x)$$

for all real x. Then

$$\lambda^{-\frac{1}{2}}\theta_3\big(0, i\lambda^{-1}\big) \leq \int_{-\infty}^{\infty} F(x)\ \mathrm{d}x, \tag{3.17}$$

and there is equality in (3.17) *if and only if $F(z) = M_\lambda(z)$.*

Remark 3.1.1. Given $\delta > 0$, using Theorem 3.1.1 and a simple change of variables, one can see that the function $z \mapsto K_{\lambda\delta^{-2}}(\delta z)$ is the unique best L^1-approximation of exponential type $\pi\delta$ for $G_\lambda(x)$. Similarly, using Theorem 3.1.2, one can check that the functions $z \mapsto L_{\lambda\delta^{-2}}(\delta z)$ and $z \mapsto M_{\lambda\delta^{-2}}(\delta z)$ are the unique extremal minorant and majorant, respectively, of exponential type $2\pi\delta$ for $G_\lambda(x)$.

The proofs of Theorems 3.1.1 and 3.1.2 are based on suitable integral representations in which the framework of the theta functions will prove itself useful. The solution of the extremal problem for the Gaussian provides a number of interesting applications that will be discussed in the last section of the chapter.

3.2 Integral Representations

Lemma 3.2.1. *Let z and w be distinct complex numbers. Then we have*

$$\begin{aligned}\frac{G_\lambda(z)-G_\lambda(w)}{z-w} &= 2\pi\lambda^{\frac{3}{2}}\int_{-\infty}^{0}\int_{-\infty}^{0} e^{-2\pi\lambda tu}G_\lambda(z-t)G_\lambda(w-u)\,\mathrm{d}u\,\mathrm{d}t \\ &\quad - 2\pi\lambda^{\frac{3}{2}}\int_{0}^{\infty}\int_{0}^{\infty} e^{-2\pi\lambda tu}G_\lambda(z-t)G_\lambda(w-u)\,\mathrm{d}u\,\mathrm{d}t.\end{aligned} \tag{3.18}$$

Proof. It suffices to prove the identity (3.18) for $\lambda = 1$, then the general case will follow from an elementary change of variables. Therefore we simplify our notation and write $G(z) = G_1(z)$. We note that $G(z)$ satisfies the identity

$$G(z)^{-1} = \int_{-\infty}^{\infty} e^{2\pi zt}G(t)\ \mathrm{d}t \tag{3.19}$$

for all complex numbers z, and the identity

$$G(z)G(w)e^{2\pi zw} = G(z-w) \tag{3.20}$$

for all pairs of complex numbers z and w. From (3.19) we get

$$\begin{aligned}\frac{G(z)-G(w)}{z-w} &= G(z)G(w)\left\{\frac{G(w)^{-1}-G(z)^{-1}}{z-w}\right\} \\ &= G(z)G(w)(z-w)^{-1}\int_{-\infty}^{\infty}\left\{e^{2\pi wt}-e^{2\pi zt}\right\}G(t)\ \mathrm{d}t.\end{aligned} \tag{3.21}$$

Then using Fubini's theorem we find that

$$
\begin{aligned}
(z-w)^{-1} & \int_{-\infty}^{\infty} \{e^{2\pi wt} - e^{2\pi zt}\} G(t) \, \mathrm{d}t \\
& = 2\pi \int_{-\infty}^{0} \left\{ \int_{t}^{0} e^{2\pi(z-w)u} \, \mathrm{d}u \right\} e^{2\pi wt} G(t) \, \mathrm{d}t \\
& \quad - 2\pi \int_{0}^{\infty} \left\{ \int_{0}^{t} e^{2\pi(z-w)u} \, \mathrm{d}u \right\} e^{2\pi wt} G(t) \, \mathrm{d}t \\
& = 2\pi \int_{-\infty}^{0} \left\{ \int_{-\infty}^{u} e^{2\pi wt} G(t) \, \mathrm{d}t \right\} e^{2\pi(z-w)u} \, \mathrm{d}u \\
& \quad - 2\pi \int_{0}^{\infty} \left\{ \int_{u}^{\infty} e^{2\pi wt} G(t) \, \mathrm{d}t \right\} e^{2\pi(z-w)u} \, \mathrm{d}u \\
& = 2\pi \int_{-\infty}^{0} \left\{ \int_{-\infty}^{0} e^{2\pi w(t+u)} G(t+u) \, \mathrm{d}t \right\} e^{2\pi(z-w)u} \, \mathrm{d}u \\
& \quad - 2\pi \int_{0}^{\infty} \left\{ \int_{0}^{\infty} e^{2\pi w(t+u)} G(t+u) \, \mathrm{d}t \right\} e^{2\pi(z-w)u} \, \mathrm{d}u \\
& = 2\pi \int_{-\infty}^{0} \int_{-\infty}^{0} e^{2\pi(wt+zu)} G(t+u) \, \mathrm{d}t \, \mathrm{d}u \\
& \quad - 2\pi \int_{0}^{\infty} \int_{0}^{\infty} e^{2\pi(wt+zu)} G(t+u) \, \mathrm{d}t \, \mathrm{d}u.
\end{aligned} \tag{3.22}
$$

Next we apply (3.20) twice and get

$$
\begin{aligned}
G(z)G(w)e^{2\pi(wt+zu)}G(t+u) &= G(z)G(w)G(u)G(t)e^{-2\pi tu + 2\pi wt + 2\pi zu} \\
&= G(z-u)G(w-t)e^{-2\pi tu}.
\end{aligned} \tag{3.23}
$$

Then we combine (3.21), (3.22) and (3.23) to obtain the special case

$$
\begin{aligned}
\frac{G(z)-G(w)}{z-w} = 2\pi & \int_{-\infty}^{0} \int_{-\infty}^{0} e^{-2\pi tu} G(z-t)G(w-u) \, \mathrm{d}u \, \mathrm{d}t \\
& - 2\pi \int_{0}^{\infty} \int_{0}^{\infty} e^{-2\pi tu} G(z-t)G(w-u) \, \mathrm{d}u \, \mathrm{d}t.
\end{aligned} \tag{3.24}
$$

The more general identity (3.18) follows by replacing z with $\lambda^{\frac{1}{2}}z$, by replacing w with $\lambda^{\frac{1}{2}}w$, and by making a corresponding change of variables in each integral on the right of (3.24). $\square$

Corollary 3.2.2. *Let z and w be distinct complex numbers. Then we have*

$$
\begin{aligned}
&\frac{G_\lambda(z)}{(z-w)^2} - \frac{G_\lambda(w)}{(z-w)^2} - \frac{G'_\lambda(w)}{z-w} \\
&= (2\pi)^2\lambda^{\frac{5}{2}} \int_{-\infty}^{0}\int_{-\infty}^{0} te^{-2\pi\lambda tu} G_\lambda(z-t)\big\{G_\lambda(w) - G_\lambda(w-u)\big\}\ \mathrm{d}u\ \mathrm{d}t \\
&\quad - (2\pi)^2\lambda^{\frac{5}{2}} \int_{0}^{\infty}\int_{0}^{\infty} te^{-2\pi\lambda tu} G_\lambda(z-t)\big\{G_\lambda(w) - G_\lambda(w-u)\big\}\ \mathrm{d}u\ \mathrm{d}t.
\end{aligned}
\tag{3.25}
$$

Proof. We differentiate both sides of (3.18) with respect to w and obtain the identity

$$
\begin{aligned}
&\frac{G_\lambda(z)}{(z-w)^2} - \frac{G_\lambda(w)}{(z-w)^2} - \frac{G'_\lambda(w)}{z-w} \\
&= 2\pi\lambda^{\frac{3}{2}} \int_{-\infty}^{0}\int_{-\infty}^{0} e^{-2\pi\lambda tu} G_\lambda(z-t) G'_\lambda(w-u)\ \mathrm{d}u\ \mathrm{d}t \\
&\quad - 2\pi\lambda^{\frac{3}{2}} \int_{0}^{\infty}\int_{0}^{\infty} e^{-2\pi\lambda tu} G_\lambda(z-t) G'_\lambda(w-u)\ \mathrm{d}u\ \mathrm{d}t.
\end{aligned}
\tag{3.26}
$$

Using integration by parts we get

$$
\begin{aligned}
&\int_{-\infty}^{0} e^{-2\pi\lambda tu} G'_\lambda(w-u)\ \mathrm{d}u \\
&\qquad = 2\pi\lambda \int_{-\infty}^{0} te^{-2\pi\lambda tu}\big\{G_\lambda(w) - G_\lambda(w-u)\big\}\ \mathrm{d}u,
\end{aligned}
\tag{3.27}
$$

and

$$
\begin{aligned}
&\int_{0}^{\infty} e^{-2\pi\lambda tu} G'_\lambda(w-u)\ \mathrm{d}u \\
&\qquad = 2\pi\lambda \int_{0}^{\infty} te^{-2\pi\lambda tu}\big\{G_\lambda(w) - G_\lambda(w-u)\big\}\ \mathrm{d}u.
\end{aligned}
\tag{3.28}
$$

The corollary follows now by combining (3.26), (3.27) and (3.28). □

In order to apply the identities (3.12), (3.13) and (3.14), we require simple estimates for certain partial sums.

Lemma 3.2.3. *For all real u and positive N we have*

$$\sum_{n=-N-1}^{N} (-1)^n G_\lambda(n+\tfrac{1}{2}-u) \ll_\lambda \min\{1,|u|\}, \tag{3.29}$$

$$\sum_{n=-N-1}^{N} \big\{G_\lambda(n+\tfrac{1}{2}) - G_\lambda(n+\tfrac{1}{2}-u)\big\} \ll_\lambda \min\{1,|u|\}, \tag{3.30}$$

$$\sum_{n=-N}^{N} \big\{G_\lambda(n) - G_\lambda(n-u)\big\} \ll_\lambda \min\{1,|u|\}, \tag{3.31}$$

where the constant implied by $\ll_\lambda$ depends on λ, but not on u or N.

Proof. For each positive integer N,

$$u \mapsto S_{\lambda,N}(u) = \sum_{n=-N-1}^{N} (-1)^n G_\lambda(n+\tfrac{1}{2}-u)$$

is an odd function of u. Hence its derivative is an even function of u. Therefore we get

$$\begin{aligned} \big|S_{\lambda,N}(u)\big| &= \left|\int_0^u S'_{\lambda,N}(v)\ \mathrm{d}v\right| \\ &\le \int_0^{|u|} \left\{\sum_{n=-\infty}^{\infty} \big|G'_\lambda(n+\tfrac{1}{2}-v)\big|\right\} \mathrm{d}v \\ &\le C_\lambda |u|, \end{aligned}$$

where

$$C_\lambda = \sup_{v\in\mathbb{R}} \left\{\sum_{n=-\infty}^{\infty} \big|G'_\lambda(n+\tfrac{1}{2}-v)\big|\right\}$$

is obviously finite. We also have

$$\big|S_{\lambda,N}(u)\big| \le \sup_{v\in\mathbb{R}} \left\{\sum_{n=-\infty}^{\infty} \big|G_\lambda(n+\tfrac{1}{2}-v)\big|\right\} < \infty,$$

and the bound (3.29) follows.

The proof of (3.30) and (3.31) is very similar. $\square$

We have noted that the entire function $z \mapsto G_\lambda(z) - K_\lambda(z)$ vanishes at each point of the coset $\mathbb{Z} + \frac{1}{2}$. It follows that

$$z \mapsto \frac{\pi}{\cos \pi z}\Big\{G_\lambda(z) - K_\lambda(z)\Big\}$$

is an entire function.

Lemma 3.2.4. *For all complex z we have*

$$\begin{aligned} &\frac{\pi}{\cos \pi z}\Big\{G_\lambda(z) - K_\lambda(z)\Big\} \\ &\qquad = \pi\lambda \int_{-\infty}^{\infty} \frac{G_\lambda(z-t)}{\cosh \pi\lambda t} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cosh 2\pi\lambda t u \ \theta_1\big(u, i\lambda^{-1}\big) \ \mathrm{d}u \ \mathrm{d}t. \end{aligned} \tag{3.32}$$

Proof. We use the partial fraction expansion

$$\lim_{N\to\infty} \sum_{n=-N-1}^{N} \frac{(-1)^{n+1}}{z - n - \frac{1}{2}} = \frac{\pi}{\cos \pi z}, \tag{3.33}$$

which converges uniformly on compact subsets of $\mathbb{C} \setminus \{\mathbb{Z} + \frac{1}{2}\}$. Then it follows from (3.3) and (3.33) that

$$\begin{aligned} &\frac{\pi}{\cos \pi z}\Big\{G_\lambda(z) - K_\lambda(z)\Big\} \\ &\qquad = \lim_{N\to\infty} \sum_{n=-N-1}^{N} (-1)^{n+1} \left\{\frac{G_\lambda(z) - G_\lambda(n+\frac{1}{2})}{z - n - \frac{1}{2}}\right\}. \end{aligned} \tag{3.34}$$

As the function on the left of (3.34) is entire and a compact subset of $\mathbb{C}$ intersects $\mathbb{Z} + \frac{1}{2}$ in finitely many points, we find that the limit on the right of (3.34) converges uniformly on compact subsets of $\mathbb{C}$.

For positive integers N and all real u let

$$S_{\lambda,N}(u) = \sum_{n=-N-1}^{N} (-1)^n G_\lambda(n + \tfrac{1}{2} - u).$$

Then (3.12) implies that

$$\lim_{N\to\infty} S_{\lambda,N}(u) = \lambda^{-\frac{1}{2}}\theta_1\big(u - \tfrac{1}{2}, i\lambda^{-1}\big). \tag{3.35}$$

We use the identity (3.18) with $w = n + \frac{1}{2}$ and sum over integers n satisfying $-N - 1 \le n \le N$. We find that

$$\begin{aligned}\sum_{n=-N-1}^{N} &(-1)^{n+1}\left\{\frac{G_\lambda(z) - G_\lambda(n+\frac{1}{2})}{z - n - \frac{1}{2}}\right\}\\ &= 2\pi\lambda^{\frac{3}{2}}\int_0^\infty\int_0^\infty e^{-2\pi\lambda tu}G_\lambda(z-t)S_{\lambda,N}(u)\ \mathrm{d}u\ \mathrm{d}t\\ &\quad - 2\pi\lambda^{\frac{3}{2}}\int_{-\infty}^0\int_{-\infty}^0 e^{-2\pi\lambda tu}G_\lambda(z-t)S_{\lambda,N}(u)\ \mathrm{d}u\ \mathrm{d}t.\end{aligned} \tag{3.36}$$

Next we let $N \to \infty$ on both sides of (3.36). The limit on the left hand side is determined by (3.34). On the right hand side we use (3.29) and the dominated convergence theorem to move the limit inside the integral. Then we use (3.35). In this way we arrive at the identity

$$\begin{aligned}\frac{\pi}{\cos\pi z}&\Big\{G_\lambda(z) - K_\lambda(z)\Big\}\\ &= 2\pi\lambda\int_0^\infty\int_0^\infty e^{-2\pi\lambda tu}G_\lambda(z-t)\theta_1\big(u - \tfrac{1}{2}, i\lambda^{-1}\big)\ \mathrm{d}u\ \mathrm{d}t\\ &\quad - 2\pi\lambda\int_{-\infty}^0\int_{-\infty}^0 e^{-2\pi\lambda tu}G_\lambda(z-t)\theta_1\big(u - \tfrac{1}{2}, i\lambda^{-1}\big)\ \mathrm{d}u\ \mathrm{d}t.\end{aligned} \tag{3.37}$$

If $0 < t$ then, using (3.10) and the fact that $u \mapsto \theta_1\big(u, i\lambda^{-1}\big)$ is an even function, we get

$$\begin{aligned}\int_0^\infty & e^{-2\pi\lambda tu}\theta_1\big(u-\tfrac{1}{2}, i\lambda^{-1}\big)\ \mathrm{d}u \\ &= \sum_{m=0}^\infty \int_0^1 e^{-2\pi\lambda t(u+m)}\theta_1\big(u+m-\tfrac{1}{2}, i\lambda^{-1}\big)\ \mathrm{d}u \\ &= \sum_{m=0}^\infty (-1)^m e^{-2\pi\lambda tm}\int_0^1 e^{-2\pi\lambda tu}\theta_1\big(u-\tfrac{1}{2}, i\lambda^{-1}\big)\ \mathrm{d}u \\ &= \big\{e^{\pi\lambda t}+e^{-\pi\lambda t}\big\}^{-1}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi\lambda tu}\theta_1\big(u, i\lambda^{-1}\big)\ \mathrm{d}u \\ &= \big\{2\cosh\pi\lambda t\big\}^{-1}\int_{-\frac{1}{2}}^{\frac{1}{2}} \cosh 2\pi\lambda tu\ \theta_1\big(u, i\lambda^{-1}\big)\ \mathrm{d}u.\end{aligned} \tag{3.38}$$

If $t < 0$ then in a similar manner we find that

$$\begin{aligned}\int_{-\infty}^0 & e^{-2\pi\lambda tu}\theta_1\big(u-\tfrac{1}{2}, i\lambda^{-1}\big)\ \mathrm{d}u \\ &= -\big\{2\cosh\pi\lambda t\big\}^{-1}\int_{-\frac{1}{2}}^{\frac{1}{2}} \cosh 2\pi\lambda tu\ \theta_1\big(u, i\lambda^{-1}\big)\ \mathrm{d}u.\end{aligned} \tag{3.39}$$

The identity (3.32) follows now by combining (3.37), (3.38) and (3.39). $\square$

Because $z \mapsto L_\lambda(z)$ interpolates both the value of $G_\lambda(z)$ and the value of its derivative $G'_\lambda(z)$ at each point of the coset $\mathbb{Z}+\frac{1}{2}$, the entire function

$$z \mapsto G_\lambda(z) - L_\lambda(z)$$

has a zero of multiplicity at least 2 at each point of $\mathbb{Z}+\frac{1}{2}$. It follows that

$$z \mapsto \left(\frac{\pi}{\cos\pi z}\right)^2\Big\{G_\lambda(z) - L_\lambda(z)\Big\}$$

is an entire function. In a similar manner, we find that

$$z \mapsto \left(\frac{\pi}{\sin\pi z}\right)^2\Big\{M_\lambda(z) - G_\lambda(z)\Big\}$$

is an entire function.

Lemma 3.2.5. *For all complex z we have*

$$\begin{aligned}&\left(\frac{\pi}{\cos \pi z}\right)^2 \Big\{G_\lambda(z) - L_\lambda(z)\Big\}\\ &= 2\pi^2\lambda^2 \int_{-\infty}^{\infty} \frac{tG_\lambda(z-t)}{\sinh \pi\lambda t} \int_{-\frac12}^{\frac12} e^{-2\pi\lambda t u}\big\{\theta_3\big(u, i\lambda^{-1}\big) - \theta_3\big(\tfrac12, i\lambda^{-1}\big)\big\}\mathrm{d}u\,\mathrm{d}t,\end{aligned} \tag{3.40}$$

and

$$\begin{aligned}&\left(\frac{\pi}{\sin \pi z}\right)^2 \Big\{M_\lambda(z) - G_\lambda(z)\Big\}\\ &= 2\pi^2\lambda^2 \int_{-\infty}^{\infty} \frac{tG_\lambda(z-t)}{\sinh \pi\lambda t} \int_{-\frac12}^{\frac12} e^{-2\pi\lambda t u}\big\{\theta_2\big(\tfrac12, i\lambda^{-1}\big) - \theta_2\big(u, i\lambda^{-1}\big)\big\}\mathrm{d}u\,\mathrm{d}t.\end{aligned} \tag{3.41}$$

Proof. In order to establish (3.40) we use the partial fraction expansion

$$\lim_{N\to\infty} \sum_{n=-N-1}^{N} \frac{1}{\big(z-n-\frac12\big)^2} = \left(\frac{\pi}{\cos \pi z}\right)^2, \tag{3.42}$$

which converges uniformly on compact subsets of $\mathbb{C}\setminus\big\{\mathbb{Z}+\frac12\big\}$. Then it follows from (3.4) and (3.42) that

$$\begin{aligned}&\left(\frac{\pi}{\cos \pi z}\right)^2 \Big\{G_\lambda(z) - L_\lambda(z)\Big\}\\ &= \lim_{N\to\infty} \sum_{n=-N-1}^{N} \left\{\frac{G_\lambda(z)}{(z-n-\frac12)^2} - \frac{G_\lambda(n+\frac12)}{(z-n-\frac12)^2} - \frac{G'_\lambda(n+\frac12)}{z-n-\frac12}\right\}.\end{aligned} \tag{3.43}$$

As in the proof of Lemma 3.2.4, the limit on the right of (3.43) converges uniformly on compact subsets of $\mathbb{C}$.

For positive integers N and all real u let

$$T_{\lambda,N}(u) = \sum_{n=-N-1}^{N} \big\{G_\lambda(n+\tfrac12) - G_\lambda(n+\tfrac12-u)\big\}.$$

From (3.13) we conclude that

$$\lim_{N\to\infty} T_{\lambda,N}(u) = \lambda^{-\frac{1}{2}}\big\{\theta_2\big(0, i\lambda^{-1}\big) - \theta_2\big(u, i\lambda^{-1}\big)\big\}. \tag{3.44}$$

We use the identity (3.25) with $w = n + \frac{1}{2}$ and sum over integers n satisfying $-N-1 \leq n \leq N$. We get

$$\begin{aligned}
&\sum_{n=-N-1}^{N} \left\{ \frac{G_\lambda(z)}{(z-n-\frac{1}{2})^2} - \frac{G_\lambda(n+\frac{1}{2})}{(z-n-\frac{1}{2})^2} - \frac{G'_\lambda(n+\frac{1}{2})}{z-n-\frac{1}{2}} \right\} \\
&\qquad = (2\pi)^2\lambda^{5/2} \int_{-\infty}^{0}\int_{-\infty}^{0} te^{-2\pi\lambda tu} G_\lambda(z-t) T_{\lambda,N}(u)\ \mathrm{d}u\ \mathrm{d}t \\
&\qquad\quad - (2\pi)^2\lambda^{5/2} \int_{0}^{\infty}\int_{0}^{\infty} te^{-2\pi\lambda tu} G_\lambda(z-t) T_{\lambda,N}(u)\ \mathrm{d}u\ \mathrm{d}t.
\end{aligned} \tag{3.45}$$

As in the proof of Lemma 3.2.4, we let $N \to \infty$ on both sides of (3.45). The limit on the left hand side is determined by (3.43). On the right hand side we use (3.30), the dominated convergence theorem and (3.44). In this way we obtain the identity

$$\begin{aligned}
&\left(\frac{\pi}{\cos\pi z}\right)^2 \Big\{ G_\lambda(z) - L_\lambda(z) \Big\} \\
&= (2\pi\lambda)^2 \int_{-\infty}^{0}\int_{-\infty}^{0} te^{-2\pi\lambda tu} G_\lambda(z-t) \big\{\theta_2\big(0, i\lambda^{-1}\big) - \theta_2\big(u, i\lambda^{-1}\big)\big\}\ \mathrm{d}u\ \mathrm{d}t \\
&\quad - (2\pi\lambda)^2 \int_{0}^{\infty}\int_{0}^{\infty} te^{-2\pi\lambda tu} G_\lambda(z-t) \big\{\theta_2\big(0, i\lambda^{-1}\big) - \theta_2\big(u, i\lambda^{-1}\big)\big\}\ \mathrm{d}u\ \mathrm{d}t.
\end{aligned} \tag{3.46}$$

If $0 < t$ then, using that $v \mapsto \theta_2(v,\tau)$ has period 1 and (3.11), we get

$$\begin{aligned}
&\int_0^\infty e^{-2\pi\lambda tu}\bigl\{\theta_2\bigl(0,i\lambda^{-1}\bigr)-\theta_2\bigl(u,i\lambda^{-1}\bigr)\bigr\}\,\mathrm{d}u\\
&\quad=\sum_{m=0}^{\infty}\int_0^1 e^{-2\pi\lambda t(u+m)}\bigl\{\theta_2\bigl(0,i\lambda^{-1}\bigr)-\theta_2\bigl(u+m,i\lambda^{-1}\bigr)\bigr\}\,\mathrm{d}u\\
&\quad=\bigl\{1-e^{-2\pi\lambda t}\bigr\}^{-1}\int_0^1 e^{-2\pi\lambda tu}\bigl\{\theta_2\bigl(0,i\lambda^{-1}\bigr)-\theta_2\bigl(u,i\lambda^{-1}\bigr)\bigr\}\,\mathrm{d}u\\
&\quad=\bigl\{2\sinh\pi\lambda t\bigr\}^{-1}\int_{-\frac12}^{\frac12} e^{-2\pi\lambda tu}\bigl\{\theta_3\bigl(\tfrac12,i\lambda^{-1}\bigr)-\theta_3\bigl(u,i\lambda^{-1}\bigr)\bigr\}\,\mathrm{d}u.
\end{aligned} \tag{3.47}$$

If $t < 0$ then in a similar manner we find that

$$\begin{aligned}
&\int_{-\infty}^0 e^{-2\pi\lambda tu}\bigl\{\theta_2\bigl(0,i\lambda^{-1}\bigr)-\theta_2\bigl(u,i\lambda^{-1}\bigr)\bigr\}\,\mathrm{d}u\\
&\quad=-\bigl\{2\sinh\pi\lambda t\bigr\}^{-1}\int_{-\frac12}^{\frac12} e^{-2\pi\lambda tu}\bigl\{\theta_3\bigl(\tfrac12,i\lambda^{-1}\bigr)-\theta_3\bigl(u,i\lambda^{-1}\bigr)\bigr\}\,\mathrm{d}u.
\end{aligned} \tag{3.48}$$

The identity (3.40) follows now by combining (3.46), (3.47) and (3.48).

The proof of (3.41) proceeds along the same lines using (3.14) and (3.31). □

Corollary 3.2.6. *For all real values of x we have*

$$0 < \left(\frac{\pi}{\cos\pi x}\right)^2\Bigl\{G_\lambda(x)-L_\lambda(x)\Bigr\}, \tag{3.49}$$

and

$$0 < \left(\frac{\pi}{\sin\pi x}\right)^2\Bigl\{M_\lambda(x)-G_\lambda(x)\Bigr\}. \tag{3.50}$$

In particular, the inequality (3.6) *holds for all real x.*

Proof. For real u the periodic function $u \mapsto \theta_3\bigl(u,i\lambda^{-1}\bigr)$ takes its maximum value at $u=0$ and its minimum value at $u=\frac12$. Therefore the function

$$t \mapsto \int_{-\frac12}^{\frac12} e^{-2\pi\lambda tu}\bigl\{\theta_3\bigl(u,i\lambda^{-1}\bigr)-\theta_3\bigl(\tfrac12,i\lambda^{-1}\bigr)\bigr\}\,\mathrm{d}u,$$

which appears in the integrand on the right of (3.40), is positive for all real values of t. This plainly verifies the inequality (3.49).

In a similar manner using (3.11), the periodic function $u \mapsto \theta_2\big(u, i\lambda^{-1}\big)$ takes its maximum value at $u = \frac{1}{2}$ and its minimum value at $u = 0$. Hence the function

$$t \mapsto \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi\lambda t u}\big\{\theta_2\big(\tfrac{1}{2}, i\lambda^{-1}\big) - \theta_2\big(u, i\lambda^{-1}\big)\big\}\ \mathrm{d}u,$$

which appears in the integrand on the right of (3.41), is positive for all real values of t. This establishes the inequality (3.50). $\square$

3.3 Proof of Theorem 3.1.1 - Best L^1-Approximation

We recall that $v \mapsto \theta_1\big(v, i\lambda^{-1}\big)$ is an entire function of the complex variable v. The product formula for this theta function [21, Chapter V, Theorem 6] provides the representation

$$\begin{aligned}\theta_1\big(v, i\lambda^{-1}\big) = 2e^{-\pi(4\lambda)^{-1}} &\prod_{l=1}^{\infty}\big(1 - e^{-2\pi\lambda^{-1}l}\big)\cos \pi v \\ &\prod_{m=1}^{\infty}\big(1 + e^{-2\pi\lambda^{-1}m + 2\pi i v}\big)\prod_{n=1}^{\infty}\big(1 + e^{-2\pi\lambda^{-1}n - 2\pi i v}\big).\end{aligned} \tag{3.51}$$

It follows from (3.51) that the only real zeros of $\theta_1\big(v, i\lambda^{-1}\big)$ are zeros of $\cos \pi v$. That is, the only real zeros are simple zeros at the points of $\mathbb{Z} + \frac{1}{2}$. Because $\theta_1\big(0, i\lambda^{-1}\big) > 0$, it follows that $\theta_1\big(u, i\lambda^{-1}\big) > 0$ for all real values of u in the open interval $-\frac{1}{2} < u < \frac{1}{2}$. This implies that the integral on the left of (3.15) is positive. Also, the integral on the right of (3.32) is positive for real values of $z = x$. Alternatively, we have

$$\frac{\pi}{\cos \pi x}\Big\{G_\lambda(x) - K_\lambda(x)\Big\} > 0$$

for all real x, and therefore

$$\operatorname{sgn}\Big\{G_\lambda(x) - K_\lambda(x)\Big\} = \operatorname{sgn}(\cos \pi x) \tag{3.52}$$

for all real x.

From the series expansion (3.7) we find that

$$\lambda^{-\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \theta_1(u, i\lambda^{-1})\ \mathrm{d}u = \frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{2n+1} \widehat{G}_\lambda(n + \tfrac{1}{2}), \tag{3.53}$$

where

$$\widehat{G}_\lambda(t) = \lambda^{-\frac{1}{2}} e^{-\pi\lambda^{-1}t^2} = \int_{-\infty}^{\infty} G_\lambda(x) e(-xt)\ \mathrm{d}x$$

is the Fourier transform of $G_\lambda(x)$ on $\mathbb{R}$. Now let $F(z)$ be an entire function of exponential type at most π. Without loss of generality we may assume that

$$\int_{-\infty}^{\infty} \Big|G_\lambda(x) - F(x)\Big|\ \mathrm{d}x < \infty.$$

It follows that F is integrable on $\mathbb{R}$ and therefore the Fourier transform

$$t \mapsto \widehat{F}(t) = \int_{-\infty}^{\infty} F(x)\, e(-tx)\ \mathrm{d}x$$

is continuous on $\mathbb{R}$, and supported on $\big[-\frac{1}{2}, \frac{1}{2}\big]$. The function

$$x \mapsto \operatorname{sgn}(\cos \pi x)$$

is periodic on $\mathbb{R}$ with period 2, and has the Fourier expansion

$$\operatorname{sgn}(\cos \pi x) = \lim_{N\to\infty} \frac{2}{\pi} \sum_{n=-N-1}^{N} \frac{(-1)^n}{2n+1}\, e\big((n + \tfrac{1}{2})x\big). \tag{3.54}$$

Moreover, the partial sums on the right of (3.54) are uniformly bounded, and therefore

$$\begin{aligned}
&\int_{-\infty}^{\infty} \operatorname{sgn}(\cos \pi x)\bigl\{G_\lambda(x) - F(x)\bigr\}\ \mathrm{d}x \\
&\qquad = \lim_{N\to\infty} \frac{2}{\pi} \sum_{n=-N-1}^{N} \frac{(-1)^n}{2n+1} \int_{-\infty}^{\infty} \bigl\{G_\lambda(x) - F(x)\bigr\} e\bigl((n+\tfrac{1}{2})x\bigr)\ \mathrm{d}x \\
&\qquad = \lim_{N\to\infty} \frac{2}{\pi} \sum_{n=-N-1}^{N} \frac{(-1)^n}{2n+1} \bigl\{\widehat{G}_\lambda(n+\tfrac{1}{2}) - \widehat{F}(n+\tfrac{1}{2})\bigr\} \\
&\qquad = \frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{2n+1} \widehat{G}_\lambda(n+\tfrac{1}{2}).
\end{aligned} \tag{3.55}$$

It is clear from (3.53) and (3.55) that

$$\lambda^{-\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \theta_1\bigl(u, i\lambda^{-1}\bigr)\ \mathrm{d}u \le \int_{-\infty}^{\infty} \bigl|G_\lambda(x) - F(x)\bigr|\ \mathrm{d}x,$$

and this verifies (3.15). Then (3.52), (3.53) and (3.55) lead to the identity

$$\begin{aligned}
\lambda^{-\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \theta_1\bigl(u, i\lambda^{-1}\bigr)\ \mathrm{d}u &= \int_{-\infty}^{\infty} \operatorname{sgn}(\cos \pi x)\bigl\{G_\lambda(x) - K_\lambda(x)\bigr\}\ \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \bigl|G_\lambda(x) - K_\lambda(x)\bigr|\ \mathrm{d}x.
\end{aligned} \tag{3.56}$$

Plainly (3.56) shows that there is equality in the inequality (3.15) in the case $F(z) = K_\lambda(z)$.

Finally, we *assume* that $F(z)$ is an entire function of exponential type at most π for which there is equality in the inequality (3.15). Then (3.53) and (3.55) imply that

$$\int_{-\infty}^{\infty} \operatorname{sgn}(\cos \pi x)\bigl\{G_\lambda(x) - F(x)\bigr\}\ \mathrm{d}x = \int_{-\infty}^{\infty} \bigl|G_\lambda(x) - F(x)\bigr|\ \mathrm{d}x. \tag{3.57}$$

As $x \mapsto G_\lambda(x) - F(x)$ is continuous, we conclude from (3.57) that

$$\operatorname{sgn}(\cos \pi x)\big\{G_\lambda(x) - F(x)\big\} = \big|G_\lambda(x) - F(x)\big|$$

for all real x. This implies that

$$G_\lambda(n + \tfrac{1}{2}) = K_\lambda(n + \tfrac{1}{2}) = F(n + \tfrac{1}{2})$$

for each integer n. Therefore

$$z \mapsto K_\lambda(z) - F(z) \tag{3.58}$$

is an integrable entire function of exponential type at most π and takes the value zero at each point of the set $\mathbb{Z} + \frac{1}{2}$. From basic interpolation theorems for entire functions of exponential type (see [70, Vol. II, p. 275]), we conclude that the entire function (3.58) is identically zero. This completes the proof of Theorem 3.1.1.

3.4 Proof of Theorem 3.1.2 - Best One-Sided Approximations

Let $F(z)$ be an entire function of exponential type at most 2π such that

$$F(x) \leq G_\lambda(x) \tag{3.59}$$

for all real x. Clearly we may assume that $x \mapsto F(x)$ is integrable on $\mathbb{R}$, for if not then (3.16) is trivial. Using [30, Lemma 4], (3.13) and (3.59), we find that

$$\begin{aligned} \int_{-\infty}^{\infty} F(x)\ \mathrm{d}x &= \lim_{N\to\infty} \sum_{n=-N}^{N} \Big(1 - \frac{|n|}{N+1}\Big) F(n+v) \\ &\le \lim_{N\to\infty} \sum_{n=-N}^{N} \Big(1 - \frac{|n|}{N+1}\Big) G_\lambda(n+v) \\ &= \lambda^{-\frac{1}{2}} \theta_2\big(\tfrac{1}{2} - v, i\lambda^{-1}\big) \end{aligned} \tag{3.60}$$

for all real v. We have already noted that $v \mapsto \theta_2\big(\frac{1}{2} - v, i\lambda^{-1}\big)$ takes its minimum value at $v = \frac{1}{2}$. Hence (3.60) implies that

$$\int_{-\infty}^{\infty} F(x)\ \mathrm{d}x \le \lambda^{-\frac{1}{2}} \theta_2\big(0, i\lambda^{-1}\big),$$

and this proves (3.16).

In Corollary 3.2.6 we proved that $F(z) = L_\lambda(z)$ satisfies the inequality (3.59) for all real x. In this special case there is equality in the inequality (3.60) when $v = \frac{1}{2}$. Thus we have

$$\int_{-\infty}^{\infty} L_\lambda(x)\ \mathrm{d}x = \lambda^{-\frac{1}{2}} \theta_2\big(0, i\lambda^{-1}\big).$$

Now *assume* that $F(z)$ is an entire function of exponential type at most 2π that satisfies (3.59) for all real x, and assume that there is equality in the inequality (3.60). It follows that $v = \frac{1}{2}$ and

$$F(n + \tfrac{1}{2}) = G_\lambda(n + \tfrac{1}{2})$$

for all integers n. Then from (3.59) we also get

$$F'(n + \tfrac{1}{2}) = G'_\lambda(n + \tfrac{1}{2})$$

for all integers n. Of course this shows that the entire function

$$z \mapsto F(z) - L_\lambda(z) \tag{3.61}$$

has exponential type at most 2π, vanishes at each point of $\mathbb{Z} + \frac{1}{2}$, and its derivative also vanishes at each point of $\mathbb{Z} + \frac{1}{2}$. By a second application of [30, Lemma 4] we conclude that the entire function (3.61) is identically zero. This proves part (i) of Theorem 3.1.2. The majorant part (ii) can be proved by the same sort of argument.

3.5 Applications

In this section we make use of the extremal approximations for the Gaussian to obtain new results in approximation theory and analytic number theory. By integrating the positive parameter λ we will be able to achieve the solution of the extremal problem for functions that were so far inaccessible, for instance the functions $f(x) = |x|^\beta$, $\beta > 0$, and a family of positive definite functions. We will also exploit the new Hilbert-type inequalities that arise from these problems, in particular proving a discrete analogue of the one dimensional Hardy-Littlewood-Sobolev inequality, and obtaining optimal bounds for the extremal eigenvalues of a positive definite matrix derived from a Gaussian.

3.5.1 Integration on the Parameter λ

For $\delta > 0$, let $\mathcal{M}_\delta(\mathbb{R}^+)$ be the collection of measures μ on the Borel subsets of $(0, \infty)$ such that

$$\int_0^\infty \left(1 + \lambda^{-\frac{1}{2}}\right) e^{-\pi\lambda^{-1}\delta^2} \mathrm{d}\mu(\lambda) < \infty. \tag{3.62}$$

Let $\mu \in \mathcal{M}_\delta(\mathbb{R}^+)$ for some $\delta > 0$. From (3.62) we have

$$\int_\epsilon^{1/\epsilon} \mathrm{d}\mu(\lambda) < \infty\,,$$

for any $0 < \epsilon < 1$. Since the measure μ restricted to the interval $[\epsilon, \frac{1}{\epsilon}]$ is finite, we can define the real function

$$g_\epsilon(x) = \int_\epsilon^{1/\epsilon} G_\lambda(x)\, \mathrm{d}\mu(\lambda)\,,$$

and an application of Fubini's theorem gives us

$$\widehat{g}_\epsilon(t) = \int_\epsilon^{1/\epsilon} \widehat{G}_\lambda(t)\, \mathrm{d}\mu(\lambda) = \int_\epsilon^{1/\epsilon} \lambda^{-\frac{1}{2}}\, e^{-\pi\lambda^{-1}t^2} \mathrm{d}\mu(\lambda). \tag{3.63}$$

Since the measure μ satisfies (3.62), the limit on the right hand side of (3.63) as $\epsilon \to 0$ is finite for $|t| \geq \delta$. We *define* the function $\widehat{g}_\mu : \mathbb{R}/(-\delta, \delta) \to [0, \infty)$ by

$$\widehat{g}_\mu(t) := \lim_{\epsilon \to 0} \widehat{g}_\epsilon(t) = \int_0^\infty \lambda^{-\frac{1}{2}}\, e^{-\pi\lambda^{-1}t^2} \mathrm{d}\mu(\lambda). \tag{3.64}$$

Let $\mathcal{S}$ denote the class of Schwarz functions and $\mathcal{S}'$ be its dual space of tempered distributions. The Fourier transform is a well defined operator $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$. Two distributions $\phi_1, \phi_2 \in \mathcal{S}'$ are equal on a subset $E \subset \mathbb{R}$ if $\phi_1(f) = \phi_2(f)$ for any Schwarz function f supported in E.

An extension of Theorem 3.1.1 is given below.

Theorem 3.5.1. *Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function in the class $\mathcal{S}'$ of tempered distributions. Suppose that the Fourier transform $\widehat{h}$ satisfies the identity*

$$\widehat{h} = \widehat{g}_\mu$$

on $E = \mathbb{R}/(-\frac{1}{2}, \frac{1}{2})$, where μ is a measure in $\mathcal{M}_{1/2}(\mathbb{R}^+)$ and $\widehat{g}_\mu$ is defined in (3.64)*. Then there exists a unique best L^1-approximation $k(z)$ of exponential type π for $h(x)$. The real entire function $k(z)$ interpolates the values of $h(x)$ at the shifted integers $\mathbb{Z} + \frac{1}{2}$ satisfying*

$$\operatorname{sgn}\{h(x) - k(x)\} = \operatorname{sgn}(\cos \pi x)$$

and

$$\int_{-\infty}^{\infty} |h(x) - k(x)| \, \mathrm{d}x = \int_0^{\infty} \lambda^{-\frac{1}{2}} \int_{\frac{1}{2}}^{\frac{1}{2}} \theta_1\big(u, i\lambda^{-1}\big) \, \mathrm{d}u \, \mathrm{d}\mu(\lambda). \tag{3.65}$$

Analogously, we have the following extended version of Theorem 3.1.2.

Theorem 3.5.2. *Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function, differentiable in $\mathbb{R}/\{0\}$, in the class $\mathcal{S}'$ of tempered distributions. Suppose that the Fourier transform $\widehat{h}$ satisfies the identity*

$$\widehat{h} = \widehat{g}_\mu$$

on $E = \mathbb{R}/(-1, 1)$, where μ is a measure in $\mathcal{M}_1(\mathbb{R}^+)$ and $\widehat{g}_\mu$ is defined in (3.64)*. Then* *(i) There exists a unique extremal minorant $l(z)$ of exponential type 2π for $h(x)$. The real entire function $l(z)$ interpolates the values of $h(x)$ at the shifted integers $\mathbb{Z}+\frac{1}{2}$ satisfying*

$$\int_{-\infty}^{\infty}\big\{h(x)-l(x)\big\}\,\mathrm{d}x=\int_0^{\infty}\lambda^{-\frac{1}{2}}\Big(1-\theta_2\big(0,i\lambda^{-1}\big)\Big)\,\mathrm{d}\mu(\lambda). \tag{3.66}$$

(ii) There exists a unique extremal majorant $m(z)$ of exponential type 2π for $h(x)$. The real entire function $m(z)$ interpolates the values of $h(x)$ at the integers $\mathbb{Z}$ satisfying

$$\int_{-\infty}^{\infty}\big\{m(x)-h(x)\big\}\,\mathrm{d}x=\int_0^{\infty}\lambda^{-\frac{1}{2}}\Big(\theta_3\big(0,i\lambda^{-1}\big)-1\Big)\,\mathrm{d}\mu(\lambda). \tag{3.67}$$

From the definition (3.9) we observe that $\big(\theta_3\big(0,i\lambda^{-1}\big)-1\big)\sim e^{-\pi\lambda^{-1}}$ as $\lambda\to 0$. Also, from the transformation (3.14), we have that $\big(\theta_3\big(0,i\lambda^{-1}\big)-1\big)\sim\lambda^{\frac{1}{2}}$ as $\lambda\to\infty$. This verifies that the condition $\mu\in\mathcal{M}_1(\mathbb{R}^+)$ given by (3.62) is equivalent to the claim that the integral on the right hand side of (3.67) be finite (and also implies that (3.66) is finite). Similarly, using (3.53) and the asymptotics for the theta functions, one can show that $\mu\in\mathcal{M}_{1/2}(\mathbb{R}^+)$ guarantees that the integral on the right hand side of (3.65) is finite.

Remark 3.5.1. Starting with a measure $\mu\in\mathcal{M}_{\delta/2}(\mathbb{R}^+)$ one can work out a general formulation of Theorem 3.5.1, in which the best L^1-approximation has exponential type $\pi\delta$. Similarly, if one starts with $\mu\in\mathcal{M}_{\delta}(\mathbb{R}^+)$ it is possible to state a general form of Theorem 3.5.2, in which the extremal minorant and majorant have exponential type $2\pi\delta$. However, for most interesting applications, it is usually simpler to work out the approximations of general type once you have the ones of type π or 2π and know the scaling properties of the function $h(x)$. Another important feature is that Theorems 3.5.1 and 3.5.2 do not provide explicit expressions for the approximations. Instead, we list the properties that matter for the applications, like the interpolation points and the value of the minimal integral. For some choices of $h(x)$, one can use the interpolation formulas described in [67, Theorems 9 and 10] to obtain explicit series expansions for the approximations.

The following two lemmas are simple to prove and will aid in proof of Theorems 3.5.1 and 3.5.2.

Lemma 3.5.3. *Let $0 < \epsilon \leq \lambda \leq 1/\epsilon$. For all $z \in \mathbb{C}$ we have*

$$|K_\lambda(z)| \ll_\epsilon (1 + |z|)\, e^{\pi|z|}\,,$$

$$|L_\lambda(z)| \ll_\epsilon (1 + |z|)\, e^{2\pi|z|}\,,$$

$$|M_\lambda(z)| \ll_\epsilon (1 + |z|)\, e^{2\pi|z|}\,,$$

where the constant implied by $\ll_\epsilon$ depends on ϵ, but not on λ or z.

Proof. It suffices to look at the series representation of $K_\lambda(z)$, $L_\lambda(z)$ and $M_\lambda(z)$ given by (3.3), (3.4) and (3.5), respectively. □

Lemma 3.5.4. *(i) Let $\mu \in \mathcal{M}_{1/2}(\mathbb{R}^+)$. The function*

$$k_\epsilon(z) = \int_\epsilon^{1/\epsilon} K_\lambda(z)\, \mathrm{d}\mu(\lambda)$$

satisfies

$$\operatorname{sgn}\{g_\epsilon(x) - k_\epsilon(x)\} = \operatorname{sgn}(\cos \pi x) \tag{3.68}$$

and is the unique best L^1-approximation of exponential type π for $g_\epsilon(x)$.

(ii) Let $\mu \in \mathcal{M}_1(\mathbb{R}^+)$. The functions

$$l_\epsilon(z) = \int_\epsilon^{\frac{1}{\epsilon}} L_\lambda(z)\,\mathrm{d}\mu(\lambda) \quad \textit{and} \quad m_\epsilon(z) = \int_\epsilon^{\frac{1}{\epsilon}} M_\lambda(z)\,\mathrm{d}\mu(\lambda)$$

are the unique extremal minorant and majorant, respectively, of exponential type 2π of $g_\epsilon(x)$.

Proof. From Lemma 3.5.3 the functions $k_\epsilon(z)$, $l_\epsilon(z)$ and $m_\epsilon(z)$ have the required exponential type, since the measure μ is finite on the interval $[\epsilon, \frac{1}{\epsilon}]$. The rest follows the arguments of the proofs of Theorems 3.1.1 and 3.1.2. □

Proof of Theorem 3.5.2. We prove the part (ii) here. Part (i) should be analogous. Define

$$d_\epsilon(x) = m_\epsilon(x) - g_\epsilon(x).$$

By Poisson summation formula we have

$$\sum_{l=-\infty}^{\infty} d_\epsilon(x+l) = \sum_{k=-\infty}^{\infty} \widehat{d_\epsilon}(k) e(kx). \tag{3.69}$$

By the Paley-Wiener theorem and the fact that $m_\epsilon(x)$ interpolates $g_\epsilon(x)$ at the integers, plugging $x = 0$ in (3.69) we obtain

$$\widehat{d_\epsilon}(0) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \widehat{g}_\epsilon(k). \tag{3.70}$$

Using (3.70) back into (3.69) we arrive at

$$\sum_{l=-\infty}^{\infty} d_\epsilon(x+l) = \sum_{k\neq 0} \widehat{g}_\epsilon(k)\big(1 - e(kx)\big). \tag{3.71}$$

For a fixed x, the sequence $d_\epsilon(x)$ is monotone increasing as $\epsilon \to 0$ and so we can define the limit function

$$d_\mu(x) := \lim_{\epsilon\to 0} d_\epsilon(x). \tag{3.72}$$

Recalling the definition (3.64), we can pass the limit as $\epsilon \to 0$ in (3.71) using monotone convergence on the left hand side and dominated convergence on the right hand side to obtain

$$p_\mu(x) := \sum_{l=-\infty}^{\infty} d_\mu(x+l) = \sum_{k\neq 0} \widehat{g}_\mu(k)\big(1 - e(kx)\big)$$

for all $x \in \mathbb{R}$. The right hand side is absolutely convergent (since $\mu \in \mathcal{M}_1(\mathbb{R}^+)$) and it follows that the nonnegative periodic function $p_\mu(x)$ is continuous. In particular, $d_\mu(x)$ is finite for all $x \in \mathbb{R}$ and uniformly bounded. From the fact that $p_\mu(0) = 0$, we conclude that $d_\mu(x)$ is continuous at every $x \in \mathbb{Z}$.

An application of the monotone convergence theorem gives us

$$\begin{aligned}\int_{-\infty}^{\infty} d_\mu(x)\,\mathrm{d}x &= \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} d_\epsilon(x)\,\mathrm{d}x = \lim_{\epsilon\to 0}\widehat{d}_\epsilon(0)\\ &= \int_0^\infty \lambda^{-\frac{1}{2}}\Big(\theta_3\big(0, i\lambda^{-1}\big) - 1\Big)\,\mathrm{d}\mu(\lambda).\end{aligned} \tag{3.73}$$

Therefore the function $d_\mu(x)$ is integrable. In an analogous way, now using dominated convergence, we obtain for $|t| \geq 1$,

$$\widehat{d}_\mu(t) = \lim_{\epsilon\to 0}\widehat{d}_\epsilon(t) = -\lim_{\epsilon\to 0}\widehat{g}_\epsilon(t) = -\widehat{g}_\mu(t).$$

We are now in position to prove part (ii). Define

$$m_\mu(x) := d_\mu(x) + h(x). \tag{3.74}$$

The function $m_\mu(x)$ is certainly in the class $\mathcal{S}'$ and its Fourier transform (tempered distribution sense) verifies, for $|t| \geq 1$,

$$\widehat{m}_\mu(t) = -\widehat{g}_\mu(t) + \widehat{h}(t) = 0.$$

Therefore $\widehat{m}_\mu$ is supported in the interval $[-1, 1]$. By the Paley-Wiener theorem for distributions [37, Theorem 1.7.7] we conclude that $m_\mu(x)$ is equal almost everywhere to the restriction to $\mathbb{R}$ of an entire function of exponential type 2π that we shall call here $m(z)$. Define

$$d(x) := m(x) - h(x).$$

From (3.74) the equality $d(x) = d_\mu(x)$ holds a.e. in $\mathbb{R}$. Since $d_\mu(x) \geq 0$ for all $x \in \mathbb{R}$ and $d(x)$ is a continuous function we obtain

$$d(x) = m(x) - h(x) \geq 0$$

for all $x \in \mathbb{R}$. From the fact that d_μ is continuous at every $l \in \mathbb{Z}$, we see that $d(x) = d_\mu(x)$ in a neighborhood of all $l \in \mathbb{Z}$. This is sufficient to conclude that $m(l) = h(l)$ for all $l \in \mathbb{Z}$.

Now suppose that $F(z)$ is a real entire function of exponential type 2π such that

$$F(x) \geq h(x) \tag{3.75}$$

for all $x \in \mathbb{R}$, and also

$$\int_{-\infty}^{\infty} \big\{F(x) - h(x)\big\}\, \mathrm{d}x < \infty.$$

If we define

$$J(z) := F(z) - m(z)\,,$$

we see that $J(z)$ is a real entire function of exponential type 2π and integrable on $\mathbb{R}$. By an application of [30, Lemma 4] we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \big\{F(x) - m(x)\big\}\,\mathrm{d}x = \widehat{J}(0) &= \lim_{N\to\infty} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) J(n) \\ &= \lim_{N\to\infty} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) \big(F(n) - h(n)\big) \geq 0.\end{aligned} \tag{3.76}$$

We conclude from (3.76) that

$$\int_{-\infty}^{\infty} \big\{F(x) - h(x)\big\}\,\mathrm{d}x \geq \int_{-\infty}^{\infty} \big\{m(x) - h(x)\big\}\,\mathrm{d}x.$$

Equality happens in (3.76) if and only if $F(n) = h(n) = m(n)$, for all $n \in \mathbb{Z}$. Since $h(x)$ is differentiable in $\mathbb{R}/\{0\}$, from (3.75) we conclude that

$$F'(n) = h'(n) = m'(n)$$

for all $n \in \mathbb{Z}$, $n \neq 0$. A new application of [30, Lemma 4] implies that $F(z) = m(z)$, for all $z \in \mathbb{C}$, thus proving the uniqueness. $\square$

Proof of Theorem 3.5.1. It follows the same outline described above, making use of the new ingredient (3.68) to keep monotonicity when necessary. We define now

$$d_\epsilon(x) = g_\epsilon(x) - k_\epsilon(x).$$

Using (3.68) we see that, for a fixed x, the sequence $d_\epsilon(x)$ is monotone and we can define the limit function

$$d_\mu(x) := \lim_{\epsilon\to 0} d_\epsilon(x).$$

The rest follows analogously using the Poisson summation formula

$$\sum_{l=-\infty}^{\infty} d_\epsilon(x+2l) = \sum_{k\in\mathbb{Z}} \frac{1}{2}\, \widehat{d}_\epsilon\left(\tfrac{k}{2}\right) e\left(\tfrac{kx}{2}\right).$$

and the ingredients of the proof of Theorem 3.1.1. □

3.5.2 Special Cases

Theorems 3.5.1 and 3.5.2 are powerful tools that allow us to solve the extremal problem for new classes of functions. The main reason for this is the fact that a measure satisfying (3.62) can have a very wild behavior near the origin.

For instance, one can consider a finite measure μ. In this case, the measure $\mu \in \mathcal{M}_\delta(\mathbb{R}^+)$ for any $\delta > 0$, and Theorems 3.5.1 and 3.5.2 provide a qualitative description of the solution of the extremal problem for the functions $\Psi : \mathbb{R} \to \mathbb{R}$ given by

$$\Psi(x) = \int_0^\infty e^{-\pi\lambda x^2} \mathrm{d}\mu(\lambda). \tag{3.77}$$

A classical result due to Schoenberg [57] establishes that a function $\Psi(x)$ admits the representation (3.77) if and only if $\Phi : \mathbb{R}^n \to \mathbb{R}$ given by $\Phi(\vec{x}) = \Psi(|\vec{x}|)$ is positive definite for all dimensions n. These functions have applications in classical analysis and data interpolation problems. Among the functions of this type we can highlight

(i) $\Psi(x) = e^{-c|x|^{2r}}$, $c \geq 0$ and $0 \leq r \leq 1$.

(ii) $\Psi(x) = (|x|^2 + c^2)^{-\beta}$, $c > 0$ and $\beta \geq 0$.

The first of these examples shows in particular that we can recover all the extremal theory for $f(x) = e^{-\lambda|x|}$, $\lambda > 0$, developed in [19] and [30] from Theorems 3.5.1 and 3.5.2.

Another interesting application arises by selecting the measure μ to be

$$\mu_\alpha(E) = \pi^{-\alpha+1}\int_E \lambda^{-\alpha}\mathrm{d}\lambda\,,$$

where $\alpha > 1$ and $E \subset (0,\infty)$ is a Borel subset. In other words,

$$\mathrm{d}\mu(\lambda) = \pi^{-\alpha+1}\lambda^{-\alpha}\mathrm{d}\lambda\,.$$

Again, this measure satisfies $\mu \in \mathcal{M}_\delta(\mathbb{R}^+)$ for any $\delta > 0$. We will make use of the following lemma.

Lemma 3.5.5. *Let $f(x) = |x|^\beta$, where $\beta > 0$. For any $\epsilon > 0$, the Fourier transform (tempered distribution sense) of this function satisfies*

$$\widehat{f}(t) = \frac{\Gamma\left(\frac{\beta+1}{2}\right)}{\pi^{\frac{2\beta+1}{2}}\Gamma\left(\frac{-\beta}{2}\right)}|t|^{-1-\beta} \tag{3.78}$$

for $|t| \geq \epsilon$.

Proof. Let φ be a Schwarz function vanishing in $B_\epsilon(0)$. A classical result in harmonic analysis [60, Lemma 1, p. 117] states

$$\int_{-\infty}^{\infty} |x|^\beta\widehat{\varphi}(x)\,\mathrm{d}x = \frac{\Gamma\left(\frac{\beta+1}{2}\right)}{\pi^{\frac{2\beta+1}{2}}\Gamma\left(\frac{-\beta}{2}\right)}\int_{-\infty}^{\infty} |x|^{-1-\beta}\varphi(x)\,\mathrm{d}x \tag{3.79}$$

for $-1 < \beta < 0$. By analytic continuation we can extend (3.79) to all $\beta \in \mathbb{C}$ with $-1 < \Re(\beta)$. This proves the lemma. $\square$

From (3.64), considering $\beta = 2\alpha - 2$, we obtain

$$\widehat{g}_{\mu_\alpha}(t) = \lim_{\epsilon \to 0} \widehat{g}_\epsilon(t) = \pi^{-\alpha+1} \int_0^\infty \lambda^{-\frac{1}{2}}\, e^{-\pi\lambda^{-1}t^2}\, \lambda^{-\alpha} \mathrm{d}\lambda = \frac{\Gamma\big(\frac{\beta+1}{2}\big)}{\pi^{\frac{2\beta+1}{2}}} |t|^{-1-\beta}. \quad (3.80)$$

Using Theorems 3.5.1 and 3.5.2 with (3.78) and (3.80) we can solve the extremal problem for the functions $f(x) = |x|^\beta$ with $\beta > 0$. This is a significant extension of previous results obtained in [18] and [19], where the cases $-1 < \beta < 1$ were solved, and [49], where the cases $\beta = 2k+1$, $k \in \mathbb{N}$ were solved.

Corollary 3.5.6. *Let $\beta > 0$, $\beta \neq 2k$, $k \in \mathbb{Z}$. Let*

$$h_\beta(x) = \Gamma\big(-\tfrac{\beta}{2}\big)|x|^\beta\,.$$

(i) There exists a unique best L^1-approximation $k_\beta(z)$ of exponential type π for $h_\beta(x)$. The entire function $k_\beta(z)$ interpolates the values of $h_\beta(x)$ at $\mathbb{Z} + \frac{1}{2}$ satisfying

$$\operatorname{sgn}\{h_\beta(x) - k_\beta(x)\} = \operatorname{sgn}(\cos \pi x)$$

and

$$\int_{-\infty}^{\infty} |h_\beta(x) - k_\beta(x)|\, \mathrm{d}x = \frac{2}{\pi^{\frac{2\beta+3}{2}}} \Gamma(\tfrac{\beta+1}{2}) \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+\frac{1}{2})^{\beta+2}}.$$

(ii) There exists a unique extremal minorant $l_\beta(z)$ of exponential type 2π for $h_\beta(x)$. The entire function $l_\beta(z)$ interpolates the values of $h_\beta(x)$ at $\mathbb{Z}+\frac{1}{2}$ and satisfies

$$\int_{-\infty}^{\infty} \{h_\beta(x) - l_\beta(x)\}\, \mathrm{d}x = \frac{2}{\pi^{\frac{2\beta+1}{2}}} \Gamma(\tfrac{\beta+1}{2})(1 - 2^{-\beta})\zeta(\beta+1).$$

(iii) There exists a unique extremal majorant $m_\beta(z)$ of exponential type 2π for $h_\beta(x)$. The entire function $m_\beta(z)$ interpolates the values of $h_\beta(x)$ at $\mathbb{Z}$ and satisfies

$$\int_{-\infty}^{\infty} \{m_\beta(x) - h_\beta(x)\}\, \mathrm{d}x = \frac{2}{\pi^{\frac{2\beta+1}{2}}} \Gamma(\tfrac{\beta+1}{2})\zeta(\beta+1).$$

Above, $\zeta(z)$ denotes the Riemann zeta function.

Remark 3.5.2. It is worth pointing out that h_β changes sign at every even integer. The function h_β agrees with its extremal majorant on $\mathbb{Z}$ and with its extremal minorant on $\mathbb{Z} + \frac{1}{2}$. Hence the function $f(x) = |x|^\beta$ agrees with its extremal majorant on $\mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$ depending on the value of β, and the change from one to the other behavior occurs when β is an even integer (in this case $|x|^\beta$ is its own extremal majorant and minorant, since it is an analytic function of exponential type zero).

3.5.3 Hilbert-type Inequalities

A classical application in this theory provides sharp inequalities for some Hermitian forms by using the extremal majorants and minorants of exponential type. The Hilbert's inequality (see [54] and [67]) is the most famous example of an inequality of this type. Other examples can be found in [18], [30] and [49].

The corresponding result for Theorem 3.5.2 is the following.

Theorem 3.5.7. *Let $h : \mathbb{R} \to \mathbb{R}$ be a function is the hypotheses of Theorem 3.5.2. Let $\{\xi_l\}_{l=1}^{L}$ be a sequence of well-spaced real numbers, i.e. $1 \leq |\xi_j - \xi_k|$*

for $j \neq k$, and let $\{a_l\}_{l=1}^{L}$ be a sequence of complex numbers. We have

$$-C_1 \sum_{l=1}^{L} |a_l|^2 \leq \sum_{\substack{j,k=1 \\ j\neq k}}^{L} a_j \overline{a_k}\, \widehat{g}_\mu(\xi_j - \xi_k) \leq C_2 \sum_{l=1}^{L} |a_l|^2 \,,$$

where

$$C_1 = \int_0^\infty \lambda^{-\frac{1}{2}} \Big(1 - \theta_2\big(0, i\lambda^{-1}\big)\Big)\, \mathrm{d}\mu(\lambda)\,,$$

and

$$C_2 = \int_0^\infty \lambda^{-\frac{1}{2}} \Big(\theta_3\big(0, i\lambda^{-1}\big) - 1\Big)\, \mathrm{d}\mu(\lambda).$$

These constants are sharp.

Proof. Recall the function $d_\mu(x) \geq 0$ defined in (3.72). We have

$$\begin{aligned}
0 &\leq \int_{-\infty}^{\infty} d_\mu(x) \left| \sum_{l=1}^{L} a_l\, e(-x\xi_l) \right|^2 \mathrm{d}x \\
&= \sum_{j=1}^{L} \sum_{k=1}^{L} a_j \overline{a_k} \int_{-\infty}^{\infty} d_\mu(x)\, e(-x(\xi_j - \xi_k))\, \mathrm{d}x \\
&= \sum_{j=1}^{L} \sum_{k=1}^{L} a_j \overline{a_k}\, \widehat{d}_\mu(\xi_j - \xi_k) = \widehat{d}_\mu(0) \sum_{l=1}^{L} |a_l|^2 - \sum_{\substack{j,k=1 \\ j\neq k}}^{L} a_j \overline{a_k}\, \widehat{g}_\mu(\xi_j - \xi_k).
\end{aligned}$$

This proves the first inequality

$$\sum_{\substack{j,k=1 \\ j\neq k}}^{L} a_j \overline{a_k}\, \widehat{g}_\mu(\xi_j - \xi_k) \leq \widehat{d}_\mu(0) \sum_{l=1}^{L} |a_l|^2 \,, \tag{3.81}$$

with $C_2 = \widehat{d}_\mu(0)$ as in (3.73). To prove that the inequality (3.81) is sharp we use the limit as $\epsilon \to 0$ in (3.70) to obtain

$$\widehat{d}_\mu(0) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \widehat{g}_\mu(k).$$

We now consider the sequences $a_j = 1$ and $\xi_j = j$, for $j = 1, 2, ..., L$. The left hand side of (3.81) turns out to be equal to

$$\sum_{\substack{k=-L \\ k\neq 0}}^{L} (L - |k|)\, \widehat{g}_\mu(k).$$

Now it is just a matter of observing that

$$\lim_{L\to\infty} \sum_{\substack{k=-L \\ k\neq 0}}^{L} \frac{(L-|k|)}{L}\, \widehat{g}_\mu(k) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \widehat{g}_\mu(k) = C_2.$$

The other inequality follows the same strategy using the extremal minorant of exponential type 2π for $h(x)$, given by Theorem 3.5.2. To prove that the inequality is sharp, one can choose sequences $a_j = (-1)^j$ and $\xi_j = j$, for $j = 1, 2, ..., L$. $\square$

An interesting application of Theorem 3.5.7 and Corollary 3.5.6 is related to the discrete Hardy-Littlewood-Sobolev inequality [34, p. 288] and it extends previous results obtained in [18, Corollary 7.2] where the case $1 < \beta < 2$ was proved, and in [49, Corollary 2] where the case $\beta = 2k$, $k \in \mathbb{N}$, was proved.

Corollary 3.5.8. *Let $\{\xi_l\}_{l=1}^{L}$ be a sequence of well-spaced real numbers, i.e. $0 < \delta \leq |\xi_j - \xi_k|$ for $j \neq k$, and let $\{a_l\}_{l=1}^{L}$ be a sequence of complex numbers. For $1 < \beta$ we have*

$$-\frac{2(1-2^{1-\beta})\zeta(\beta)}{\delta^\beta} \sum_{l=1}^{L} |a_l|^2 \leq \sum_{\substack{j,k=1 \\ j\neq k}}^{L} \frac{a_j \overline{a_k}}{|\xi_j - \xi_k|^\beta} \leq \frac{2\zeta(\beta)}{\delta^\beta} \sum_{l=1}^{L} |a_l|^2\,,$$

where $\zeta(z)$ is the Riemann zeta function. The constants appearing are sharp.

As another application we highlight the Hilbert-type inequality derived directly from Theorem 3.1.2 (or from Theorem 3.5.7 with $\mu = \delta(\lambda - \lambda_0)$). It provides optimal bounds for the lowest and the largest eigenvalues of the positive definite matrix derived from $f(x) = e^{-\pi\lambda x^2}$, $\lambda > 0$.

Corollary 3.5.9. *Let $\{\xi_l\}_{l=1}^L$ be a sequence of well-spaced real numbers, i.e. $0 < \delta \leq |\xi_j - \xi_k|$ for $j \neq k$, and let $\{a_l\}_{l=1}^L$ be a sequence of complex numbers. Then*

$$C_1 \sum_{l=1}^{L} |a_l|^2 \leq \sum_{j,k=1}^{L} a_j \overline{a_k}\, e^{-\pi\lambda|\xi_j-\xi_k|^2} \leq C_2 \sum_{l=1}^{L} |a_l|^2 \,,$$

where

$$C_1 = \theta_2(0, i\lambda\delta^2)\,,$$

and

$$C_2 = \theta_3(0, i\lambda\delta^2)\,.$$

In other words, if $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_L$ are the eigenvalues of the positive definite matrix $(A_{jk}) = e^{-\pi\lambda|\xi_j-\xi_k|^2}$, then

$$C_1 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_L \leq C_2.$$

The constants C_1 and C_2 are optimal.

Chapter 4

Convolution Inequalities for the Boltzmann Collision Operator

4.1 Preliminaries

4.1.1 The Boltzmann Equation

Let us assume that we have a large space filled with particles that are considered as mass points. Assume that these particles are interacting with a specific law and that the particles are not influenced by external forces. A good model to represent such dynamical system is given by the equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f) \ \text{ in } \ (0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n. \tag{4.1}$$

The function $f(t, x, v)$, where $(t, x, v) \in (0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n$, represents the phase space density of particles which at time t and point x move with velocity v. The physical meaning implies that

$$f(t, x, v) \geq 0.$$

Equation (4.1) was derived by the first time by L. Boltzmann in 1872 in his studies of dilute gases. The term $Q(f, f)$ is known as the Boltzmann collision operator and its purpose is to model the interaction of the particles. It is customary to split this operator in two, a positive and a negative part, which quantify the appearance and disappearance of particles in *space-velocity* at a given time t. Thus, for any suitable measurable f and g, we write

$$Q(f,g) := Q^{+}(f,g) - Q^{-}(f,g)\,,$$

where

$$Q^{+}(f,g)(v) := \int_{\mathbb{R}^n}\int_{S^{n-1}} f(v')g(v'_*)B(|u|,\hat{u}\cdot\omega)\,\mathrm{d}\omega\,\mathrm{d}v_*\,, \tag{4.2}$$

and

$$Q^{-}(f,g)(v) := \int_{\mathbb{R}^n}\int_{S^{n-1}} f(v)g(v_*)B(|u|,\hat{u}\cdot\omega)\,\mathrm{d}\omega\,\mathrm{d}v_*. \tag{4.3}$$

The pair of symbols $\{v', v'_*\}$ represents the final velocities of two particles after interacting with initial velocities $\{v, v_*\}$. The relation between these is given by the formulas

$$v' = V + \frac{|u|}{2}\omega \quad \text{and} \quad v'_* = V - \frac{|u|}{2}\omega\,,$$

where V is the velocity of the center of mass of the particles, and u is the relative velocity between them, i.e.

$$V := \frac{v+v_*}{2} \quad \text{and} \quad u := v - v_*.$$

The symbol $\hat{u}$ represents the unitary vector in the direction of u ($\hat{u} = u/|u|$) and $\mathrm{d}\omega$ is the surface measure on the sphere S^{n-1}. The nature of the interactions modeled by Q^{+} is encoded in the collision kernel $B(|u|,\hat{u}\cdot\omega)$, and many physical models accept the representation (henceforth assumed)

$$B(|u|,\hat{u}\cdot\omega) = |u|^{\lambda}\, b(\hat{u}\cdot\omega) \quad \text{with} \quad -n < \lambda.$$

Depending on the parameter λ the interaction receives different names: soft-potentials when $-n < \lambda < 0$; Maxwell molecules when $\lambda = 0$; hard-potentials when $\lambda > 0$. For the (nonnegative) angular kernel $b(\hat{u} \cdot \omega)$ we will require the Grad's cut-off assumption

$$\int_{S^{n-1}} b(\hat{u} \cdot \omega) \,\mathrm{d}\omega < \infty.$$

The operator we just described is the *elastic* Boltzmann collision operator and we refer the reader to [22] for a more detailed description. For simplicity, throughout this whole chapter we will be dealing only with elastic collisions. One can also consider *inelastic* interactions (see [12] and [27]) in which a restitution coefficient is introduced in the post-collisional velocities. All the results in this chapter can also be extended to the inelastic setting. The details are in [2].

4.1.2 Convolution Inequalities

The purpose of this chapter is to present the L^p-analysis of the operator Q^+ (the hardest of the components of the collision operator Q) in the elastic case, showing that it essentially behaves as a convolution operator. The first step towards this goal is an equivalent dual definition of Q^+ given by

$$\int_{\mathbb{R}^n} Q^+(f,g)(v)\psi(v) \,\mathrm{d}v = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(v)g(v_*) \int_{S^{n-1}} \psi(v')B(|u|, \hat{u} \cdot \omega) \,\mathrm{d}\omega \,\mathrm{d}v_* \,\mathrm{d}v, \tag{4.4}$$

One can prove (4.4) starting from the original definition (4.2) and performing some changes of variables. In fact, from (4.2), we obtain

$$\int_{\mathbb{R}^n} Q^+(f,g)(v)\psi(v) \,\mathrm{d}v = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{S^{n-1}} f(v')g(v'_*)B(|u|, \hat{u} \cdot \omega)\psi(v) \,\mathrm{d}\omega \,\mathrm{d}v_* \,\mathrm{d}v.$$

Introducing the variable $u = v - v_*$ we have

$$= \int_{S^{n-1}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(v - \tfrac{1}{2}(u - |u|\omega))g(v - \tfrac{1}{2}(u + |u|\omega))B(|u|, \hat{u} \cdot \omega)\psi(v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\omega.$$

We now change to radial variables $u = |u|\sigma$ to obtain

$$= \int_{S^{n-1}} \int_{\mathbb{R}^n} \int_{S^{n-1}} \int_{\mathbb{R}^+} f(v - \tfrac{1}{2}(|u|\,\sigma - |u|\,\omega))g(v - \tfrac{1}{2}(|u|\,\sigma + |u|\,\omega))$$
$$B(|u|, \sigma \cdot \omega)\psi(v)\,|u|^{n-1}\,\mathrm{d}|u|\,\mathrm{d}\sigma\,\mathrm{d}v\,\mathrm{d}\omega.$$

Then we come back from the radial variables using $u = |u|\,\omega$,

$$= \int_{S^{n-1}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(v + \tfrac{1}{2}(u - |u|\,\sigma))g(v - \tfrac{1}{2}(u + |u|\,\sigma))B(|u|, \sigma \cdot \hat{u})\psi(v)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\sigma.$$

Let $x = v + \frac{1}{2}(u - |u|\,\sigma)$ to obtain

$$= \int_{S^{n-1}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)g(x - u)B(|u|, \sigma \cdot \hat{u})\psi(x - \tfrac{1}{2}(u - |u|\,\sigma))\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}\sigma.$$

Finally, for fixed x and σ, let $x_* = x - u$ to get

$$= \int_{S^{n-1}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)g(x_*)B(|u|, \sigma \cdot \hat{u})\psi(x')\,\mathrm{d}x\,\mathrm{d}x_*\,\mathrm{d}\sigma,$$

which is exactly (4.4) after a relabeling.

Let ψ and ϕ be bounded and continuous functions. Define the bilinear operator

$$\mathcal{P}(\psi, \phi)(u) := \int_{S^{n-1}} \psi(u^-)\phi(u^+)b(\hat{u} \cdot \omega)\,\mathrm{d}\omega\,, \tag{4.5}$$

where the symbols u^+ and u^-, commonly known as Bobylev's variables, are defined by

$$u^- := \tfrac{1}{2}(u - |u|\omega) \quad \text{and} \quad u^+ := u - u^- = \tfrac{1}{2}(u + |u|\omega). \tag{4.6}$$

The operator (4.5) was first introduced by Bobylev in a slightly different setting. Indeed, in [10] and [11] Bobylev shows that in the elastic Maxwell molecules case ($\lambda = 0$ in the collision kernel), we have

$$\widehat{Q^+(f,g)} = \mathcal{P}(\hat{f},\hat{g})\,. \tag{4.7}$$

In [22] one can find a complete presentation of the use of the Fourier transform in the analysis of the Boltzmann collision operator, including the explicit computation of the relation (4.7).

From equations (4.4) and (4.5) we obtain the following relation between the operators Q^+ and $\mathcal{P}$

$$\int_{\mathbb{R}^n} Q^+(f,g)(v)\psi(v)\,\mathrm{d}v = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)g(v-u)\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)\;|u|^\lambda\,\mathrm{d}u\,\mathrm{d}v, \tag{4.8}$$

where τ and $\mathcal{R}$ are the translation and reflection operators

$$\tau_v\psi(x) := \psi(x-v) \quad\text{and}\quad \mathcal{R}\psi(x) := \psi(-x).$$

Representation (4.8) shows that the integrability properties of the collision operator Q^+ are closely related to those of the bilinear operator $\mathcal{P}$. A similar approach is carried out in [28] which relates the operator Q^+ to a slightly different angular averaging operator.

This chapter is organized as follows. In Section 4.2 we develop the L^p-analysis of the operator $\mathcal{P}$, exploiting a symmetrization method introduced in [1] that will provide sharp constants in some of our inequalities. Generally, the constants appearing in this chapter will depend on (explicit) integral conditions on the angular collision kernel.

In Section 4.3 we prove a full Young's inequality for hard potentials. For this, consider the weighted Lebesgue spaces $L^p_k(\mathbb{R}^n)$ ($p \geq 1$, $k \geq 0$) defined by the norm

$$\|f\|_{L^p_k(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(v)|^p \, \left(1 + |v|^{pk}\right) \mathrm{d}v \right)^{1/p}.$$

We prove the following.

Theorem 4.1.1. *Let $1 \leq p,\, q,\, r \leq \infty$ with $1/p + 1/q = 1 + 1/r$. Assume that*

$$B(|u|, \hat{u} \cdot \omega) = |u|^{\lambda} b(\hat{u} \cdot \omega)\,,$$

with $\lambda \geq 0$. For $\alpha \geq 0$, the bilinear operator Q^+ extends to a bounded operator from $L^p_{\alpha+\lambda}(\mathbb{R}^n) \times L^q_{\alpha+\lambda}(\mathbb{R}^n) \to L^r_{\alpha}(\mathbb{R}^n)$ via the estimate

$$\left\|Q^+(f,g)\right\|_{L^r_\alpha(\mathbb{R}^n)} \leq C \, \|f\|_{L^p_{\alpha+\lambda}(\mathbb{R}^n)} \, \|g\|_{L^q_{\alpha+\lambda}(\mathbb{R}^n)}. \tag{4.9}$$

Young-type inequalities reveal the convolution nature of the operator Q^+ and were first introduced in the work of Gustafsson [32] for a slightly different truncated form of the collision operator. In the general formulation (4.9), the case $(p, q, r) = (p, 1, p)$ appears in the works of Mouhot-Villani [56, Theorem 2.1] and Gamba-Panferov-Villani [28, Lemma 4.1] in the study of the regularity and asymptotic Gaussian bounds for the solutions of the Boltzmann equation. The advantage of our method relies on the fact that we provide (a) simpler proofs; (b) extensions to the full range of exponents p, q, r; (c) constants depending on integral conditions on the angular kernel, rather than the classical assumptions that the kernel b vanishes near the endpoints, as in [56].

In Section 4.4 we turn our attention to study the analogues of the Young's inequality for the case of soft potentials, an issue that has never been previously discussed in the literature. It turns out that the convolution character of $Q^+(f,g)$ is still present in this case, and we basically establish that it behaves as $f * g * |u|^\lambda$, by proving the following Hardy-Littlewood-Sobolev inequality.

Theorem 4.1.2. *Let $1 < p,\ q,\ r < \infty$ with $-n < \lambda < 0$ and $1/p + 1/q = 1 + \lambda/n + 1/r$. For the kernel*

$$B(|u|, \hat{u}\cdot\omega) = |u|^\lambda\ b(\hat{u}\cdot\omega),$$

the bilinear operator Q^+ extends to a bounded operator from $L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ via the estimate

$$\left\|Q^+(f,g)\right\|_{L^r(\mathbb{R}^n)} \leq C\,\|f\|_{L^p(\mathbb{R}^n)}\,\|g\|_{L^q(\mathbb{R}^n)}. \tag{4.10}$$

The constants that we obtain for the two inequalities above are explicit, but generally not sharp. Only in the cases $\alpha = \lambda = 0$, $(p,q,r) = (2,1,2)$ and $(p,q,r) = (1,2,2)$ we find the sharp constant for the Young's inequality (4.9) (see the remark after Theorem 4.2.3). In fact, the quest for the sharp forms of these inequalities in the other cases, which could be seen as analogues of the remarkable works of Beckner [7] and Lieb [46], seems inaccessible at this time.

4.2 Radial Symmetrization Techniques

Let $G = SO(n)$ be the group of rotations of $\mathbb{R}^n$ (orthonormal transformations of determinant 1), in which we will use the variable R to designate a generic rotation. We assume that the Haar measure $\mathrm{d}\mu$ of this compact topological group is normalized so that

$$\int_G \mathrm{d}\mu(R) = 1.$$

Let $f \in L^p(\mathbb{R}^n)$, $p \geq 1$. We define the radial symmetrization $f_p^\star$ by

$$f_p^\star(x) = \left(\int_G |f(Rx)|^p \, \mathrm{d}\mu(R) \right)^{\frac{1}{p}}, \quad \text{if} \;\; 1 \leq p < \infty. \tag{4.11}$$

and

$$f_\infty^\star(x) = \text{ess sup}_{|y|=|x|} |f(y)| \tag{4.12}$$

where the essential supremum in (4.12) is taken over the sphere of radius $|x|$ with respect to the surface measure over this sphere. The symmetrization $f_p^\star$ defined in (4.11)-(4.12) can be seen as an L^p-average of f over all the rotations $R \in G$ and it satisfies the following properties:

(i) $f_p^\star$ is radial.

(ii) If f is continuous (or compactly supported) then $f_p^\star$ is also continuous (or compactly supported).

(iii) If g is a radial function then $(fg)_p^\star(x) = f_p^\star(x) g(x)$.

(iv) Let $\mathrm{d}\nu$ be a rotationally invariant measure on $\mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}\nu(x) = \int_{\mathbb{R}^n} |f_p^\star(x)|^p \, \mathrm{d}\nu(x).$$

In particular,

$$\|f\|_{L^p(\mathbb{R}^n)} = \|f_p^\star\|_{L^p(\mathbb{R}^n)}.$$

Our first result of this section is the following.

Lemma 4.2.1 (Symmetrization Lemma). *Let $f, g, \psi \in C_0(\mathbb{R}^n)$ and $1/p + 1/q + 1/r = 1$, with $1 \leq p, q, r \leq \infty$. Then*

$$\left|\int_{\mathbb{R}^n} \mathcal{P}(f,g)(u)\psi(u)\,\mathrm{d}u\right| \leq \int_{\mathbb{R}^n} \mathcal{P}(f_p^\star, g_q^\star)(u)\psi_r^\star(u)\,\mathrm{d}u.$$

Proof. From (4.5) and (4.6) we observe that for any rotation R one has

$$\mathcal{P}(f,g)(Ru) = \mathcal{P}(f \circ R, g \circ R)(u).$$

Therefore,

$$\begin{aligned} \left|\int_{\mathbb{R}^n} \mathcal{P}(f,g)(u)\,\psi(u)\,\mathrm{d}u\right| &= \left|\int_{\mathbb{R}^n} \mathcal{P}(f,g)(Ru)\,\psi(Ru)\,\mathrm{d}u\right| \\ &= \left|\int_{\mathbb{R}^n} \mathcal{P}(f \circ R, g \circ R)(u)\,\psi(Ru)\,\mathrm{d}u\right| \\ &\leq \int_{\mathbb{R}^n}\int_{S^{n-1}} |f(Ru^-)|\,|g(Ru^+)|\,|\psi(Ru)|\,b(\hat{u}\cdot\omega)\,\mathrm{d}\omega\,\mathrm{d}u. \end{aligned} \tag{4.13}$$

Note that the left hand side of (4.13) is independent of R. Thus, an integration over the group $G = SO(n)$ leads to

$$\begin{aligned} &\left|\int_{\mathbb{R}^n} \mathcal{P}(f,g)(u)\psi(u)\,\mathrm{d}u\right| \\ &\leq \int_{\mathbb{R}^n}\int_{S^{n-1}} \left(\int_G |f(Ru^-)||g(Ru^+)||\psi(Ru)|\,\mathrm{d}\mu(R)\right) b(\hat{u}\cdot\omega)\,\mathrm{d}\omega\,\mathrm{d}u. \end{aligned} \tag{4.14}$$

An application of Hölder's inequality with exponents p, q and r yields

$$\int_G |f(Ru^-)|\,|g(Ru^+)|\,|\psi(Ru)|\,\mathrm{d}\mu(R) \leq f_p^\star(u^-)\,g_q^\star(u^+)\,\psi_r^\star(u),$$

which together with equation (4.14) proves the lemma. □

The Symmetrization Lemma 4.2.1 shows that L^p-estimates for the operator $\mathcal{P}$ will follow by considering radial functions. If $f : \mathbb{R}^n \to \mathbb{R}$ is radial, we define the function $\tilde{f} : \mathbb{R}^+ \to \mathbb{R}$ by

$$f(x) = \tilde{f}(|x|).$$

In addition, for any $p \geq 1$ and $\alpha \in \mathbb{R}$ we have

$$\int_{\mathbb{R}^n} f(x)^p \, |x|^\alpha \, \mathrm{d}x = \left|S^{n-1}\right| \int_0^\infty \tilde{f}(t)^p \, t^{n-1+\alpha} \, \mathrm{d}t. \tag{4.15}$$

Hence, if we define the measure ν_α on $\mathbb{R}^n$ by

$$\mathrm{d}\nu_\alpha(x) = |x|^\alpha \mathrm{d}x \,,$$

and the measure σ_n^α on $\mathbb{R}^+$ by

$$\mathrm{d}\sigma_n^\alpha(t) = t^{n-1+\alpha} \mathrm{d}t \,,$$

equation (4.15) translates to

$$||f||_{L^p(\mathbb{R}^n,\, \mathrm{d}\nu_\alpha)} = \left|S^{n-1}\right|^{\frac{1}{p}} \; ||\tilde{f}||_{L^p(\mathbb{R}^+,\, \mathrm{d}\sigma_n^\alpha)}. \tag{4.16}$$

In the following computation we show how the operator $\mathcal{P}$ simplifies to a 1-dimensional operator when applied to radial functions. If f and g are radial, then

$$\begin{aligned} \mathcal{P}(f,g)(u) &= \int_{S^{n-1}} \tilde{f}\left(|u^-|\right) \tilde{g}\left(|u^+|\right) \, b(\hat{u} \cdot \omega) \, \mathrm{d}\omega \\ &= \int_{S^{n-1}} \tilde{f}\big(a_1(|u|, \hat{u} \cdot \omega)\big) \tilde{g}\big(a_2(|u|, \hat{u} \cdot \omega)\big) \, b(\hat{u} \cdot \omega) \, \mathrm{d}\omega \\ &= \left|S^{n-2}\right| \int_{-1}^1 \tilde{f}\big(a_1(|u|, s)\big) \tilde{g}\big(a_2(|u|, s)\big) b(s) \, (1 - s^2)^{\frac{n-3}{2}} \, \mathrm{d}s. \end{aligned} \tag{4.17}$$

The functions a_1 and a_2 are defined on $\mathbb{R}^+ \times [-1,1] \to \mathbb{R}^+$ by

$$a_1(x,s) = x\left(\tfrac{1-s}{2}\right)^{1/2} \quad \text{and} \quad a_2(x,s) = x\left(\tfrac{1+s}{2}\right)^{1/2}.$$

We conclude from (4.17) that

$$\widetilde{\mathcal{P}(f,g)}(x) = \left|S^{n-2}\right| \int_{-1}^{1} \tilde{f}\left(a_1(x,s)\right)\, \tilde{g}\left(a_2(x,s)\right)\, \mathrm{d}\xi_n^b(s)\,, \tag{4.18}$$

where the measure ξ_n^b on $[-1,1]$ is defined as

$$\mathrm{d}\xi_n^b(s) = b(s)(1-s^2)^{\frac{n-3}{2}}\,\mathrm{d}s\,.$$

In virtue of equation (4.18) we define the following bilinear operator for any two bounded and continuous functions $f, g : \mathbb{R}^+ \to \mathbb{R}$,

$$\mathcal{B}(f,g)(x) := \int_{-1}^{1} f\left(a_1(x,s)\right)\, g\left(a_2(x,s)\right) \mathrm{d}\xi_n^b(s).$$

For this operator we have the following bound.

Lemma 4.2.2. *Let $1 \le p,q,r \le \infty$ with $1/p+1/q = 1/r$. For $f \in L^p(\mathbb{R}^+, \mathrm{d}\sigma_n^\alpha)$ and $g \in L^q(\mathbb{R}^+, \mathrm{d}\sigma_n^\alpha)$ we have*

$$\|\mathcal{B}(f,g)\|_{L^r(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)} \le C\, \|f\|_{L^p(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)}\, \|g\|_{L^q(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)}\,, \tag{4.19}$$

where the sharp constant C is given by

$$C(n,\alpha,p,q,b) = \int_{-1}^{1} \left(\tfrac{1-s}{2}\right)^{-\frac{n+\alpha}{2p}} \left(\tfrac{1+s}{2}\right)^{-\frac{n+\alpha}{2q}} \mathrm{d}\xi_n^b(s). \tag{4.20}$$

Proof. Using Minkowski's inequality and Hölder's inequality with exponents p/r and q/r we obtain

$$\begin{aligned}\left\|\mathcal{B}(f,g)\right\|_{L^r(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)} &\le \int_{-1}^{1}\left(\int_0^\infty |f(a_1(x,s))|^r\,|g(a_2(x,s))|^r\,\mathrm{d}\sigma_n^\alpha(x)\right)^{\frac{1}{r}}\mathrm{d}\xi_n^b(s)\\ &\le \int_{-1}^{1}\left(\int_0^\infty |f(a_1(x,s))|^p\,\mathrm{d}\sigma_n^\alpha(x)\right)^{\frac{1}{p}}\left(\int_0^\infty |g(a_2(x,s))|^q\,\mathrm{d}\sigma_n^\alpha(x)\right)^{\frac{1}{q}}\mathrm{d}\xi_n^b(s).\end{aligned}$$

Changing the variables $y=a_1(x,s)$ we obtain

$$\left(\int_0^\infty |f(a_1(x,s))|^p\,\mathrm{d}\sigma_n^\alpha(x)\right)^{\frac{1}{p}} = \left(\tfrac{1-s}{2}\right)^{-\frac{n+\alpha}{2p}}\|f\|_{L^p(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)}\,.$$

Analogously, changing the variables $y=a_2(x,s)$, we arrive at

$$\left(\int_0^\infty |g(a_2(x,s))|^q\,\mathrm{d}\sigma_n^\alpha(x)\right)^{\frac{1}{q}} = \left(\tfrac{1+s}{2}\right)^{-\frac{n+\alpha}{2q}}\|g\|_{L^q(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)}\,.$$

This gives (4.19) with constant (4.20). To prove that this constant is indeed the best possible, one can consider the sequences $\{f_\epsilon\}$ and $\{g_\epsilon\}$ with $\epsilon>0$ defined by

$$f_\epsilon(x)=\begin{cases}\epsilon^{1/p}\,x^{-(n+\alpha-\epsilon)/p} & \text{for}\ \ 0<x<1\,,\\ 0 & \text{otherwise.}\end{cases}$$

and

$$g_\epsilon(x)=\begin{cases}\epsilon^{1/q}\,x^{-(n+\alpha-\epsilon)/q} & \text{for}\ \ 0<x<1\,,\\ 0 & \text{otherwise.}\end{cases}$$

Clearly,

$$\|f_\epsilon\|_{L^p(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)} = \|g_\epsilon\|_{L^q(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)} = 1\,,$$

and one can check that

$$\|\mathcal{B}(f_\epsilon,g_\epsilon)\|_{L^r(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)} \to C\,,$$

as $\epsilon \to 0$, where C is the constant defined in (4.20). The detailed argument is outlined in [1]. $\square$

From the Symmetrization Lemma 4.2.1 we have

$$\|\mathcal{P}(f,g)\|_{L^r(\mathbb{R}^n,\,\mathrm{d}\nu_\alpha)} \leq \|\mathcal{P}(f_p^\star, g_q^\star)\|_{L^r(\mathbb{R}^n,\,\mathrm{d}\nu_\alpha)}\,,$$

where $1/p + 1/q = 1/r$. Using equations (4.16), (4.18) and Lemma 4.2.2 we obtain

$$\begin{aligned}
\|\mathcal{P}(f_p^\star, g_q^\star)\|_{L^r(\mathbb{R}^n,\,\mathrm{d}\nu_\alpha)} &= \left|S^{n-1}\right|^{\frac{1}{r}} \ \left\|\widetilde{\mathcal{P}(f_p^\star, g_q^\star)}\right\|_{L^r(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)} \\
&= \left|S^{n-1}\right|^{\frac{1}{r}} \ \left|S^{n-2}\right| \ \|\mathcal{B}(\tilde{f}_p^\star, \tilde{g_q^\star})\|_{L^r(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)} \\
&\leq C \ \left|S^{n-1}\right|^{\frac{1}{r}} \ \left|S^{n-2}\right| \ \|\tilde{f}_p^\star\|_{L^p(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)} \ \|\tilde{g_q^\star}\|_{L^q(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)} \\
&= C \ \left|S^{n-2}\right| \ \|f\|_{L^p(\mathbb{R}^n,\,\mathrm{d}\nu_\alpha)} \ \|g\|_{L^q(\mathbb{R}^n,\,\mathrm{d}\nu_\alpha)}\,,
\end{aligned}$$

and thus we have proved the following result.

Theorem 4.2.3. *Let $1 \leq p,q,r \leq \infty$ with $1/p + 1/q = 1/r$, and $\alpha \in \mathbb{R}$. The bilinear operator $\mathcal{P}$ extends to a bounded operator from $L^p(\mathbb{R}^n, \mathrm{d}\nu_\alpha) \times L^q(\mathbb{R}^n, \mathrm{d}\nu_\alpha)$ to $L^r(\mathbb{R}^n, \mathrm{d}\nu_\alpha)$ via the estimate*

$$\|\mathcal{P}(f,g)\|_{L^r(\mathbb{R}^n,\,\mathrm{d}\nu_\alpha)} \leq C \ \|f\|_{L^p(\mathbb{R}^n,\,\mathrm{d}\nu_\alpha)} \, \|g\|_{L^q(\mathbb{R}^n,\,\mathrm{d}\nu_\alpha)}\,.$$

The sharp constant C is given by

$$C = \left|S^{n-2}\right| \int_{-1}^{1} \left(\tfrac{1-s}{2}\right)^{-\frac{n+\alpha}{2p}} \left(\tfrac{1+s}{2}\right)^{-\frac{n+\alpha}{2q}} \,\mathrm{d}\xi_n^b(s).$$

Remark 4.2.1. A simple application of Theorem 4.2.3 provides a sharp estimate for the L^2-norm in the case of Maxwell molecules. Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$. Then

$$\begin{aligned} \left\|Q^+(f,g)\right\|_{L^2(\mathbb{R}^n)} &= \left\|\widehat{Q^+(f,g)}\right\|_{L^2(\mathbb{R}^n)} = \left\|\mathcal{P}(\hat{f},\hat{g})\right\|_{L^2(\mathbb{R}^n)} \\ &\leq C_0 \,\|\hat{f}\|_{L^\infty(\mathbb{R}^n)} \,\|\hat{g}\|_{L^2(\mathbb{R}^n)} \leq C_0 \,\|f\|_{L^1(\mathbb{R}^n)} \,\|g\|_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{4.21}$$

The constant is given by

$$C_0 = \left|S^{n-2}\right| \int_{-1}^{1} \left(\tfrac{1+s}{2}\right)^{-\frac{n}{4}} \mathrm{d}\xi_n^b(s)\,.$$

Similarly, for $f \in L^2(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$ we have

$$\left\|Q^+(f,g)\right\|_{L^2(\mathbb{R}^n)} \leq C_1 \,\|f\|_{L^2(\mathbb{R}^n)} \,\|g\|_{L^1(\mathbb{R}^n)}, \tag{4.22}$$

where

$$C_1 = \left|S^{n-2}\right| \int_{-1}^{1} \left(\tfrac{1-s}{2}\right)^{-\frac{n}{4}} \mathrm{d}\xi_n^b(s).$$

To guarantee that C_0 is indeed the sharp constant in the inequality (4.21) we need approximating sequences $\widetilde{\hat{f}}_\epsilon$ and $\widetilde{\hat{g}}_\epsilon$ slightly different from those presented in the end of the proof of Lemma 4.2.2, since we would like to impose the additional constraint $f \geq 0$ to have $\|\hat{f}\|_{L^\infty(\mathbb{R}^n)} = \|f\|_{L^1(\mathbb{R}^n)}$. Heuristically, this can be done by considering $f = \delta(x)$ the Dirac delta and so $\hat{f} \equiv 1$. In practice we should choose f_ϵ a Gaussian approximation of the identity by putting

$$\widetilde{\hat{f}}_\epsilon(x) = e^{-\pi\epsilon^2 x^2}\,,$$

and

$$\widetilde{\hat{g}}_\epsilon(x) = \begin{cases} \epsilon^{1/2}\, x^{-(n-\epsilon)/2} & \text{for } \; 0 < x < 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

A similar consideration applies to the inequality (4.22). Inequalities (4.21) and (4.22) are particular cases of the Young's inequality for Q^+ that will be treated in the next section. These are the only cases where we are able to explicitly find the sharp constant.

4.3 Proof of Theorem 4.1.1 - Young's Inequality for Hard Potentials

The goal of this section is to prove Theorem 4.1.1. First we treat the case $\alpha = \lambda = 0$. The main idea is to use the relation (4.8) that establishes a connection between the operators Q^+ and $\mathcal{P}$, and then use the knowledge from the previous section. From (4.8) we have

$$I := \int_{\mathbb{R}^n} Q^+(f,g)(v)\psi(v)\,\mathrm{d}v = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)g(v-u)\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)\,\mathrm{d}u\,\mathrm{d}v.$$

The exponents p, q, r in Theorem 4.1.1 satisfy $1/p' + 1/q' + 1/r = 1$, and thus we can regroup the terms conveniently and use Hölder's inequality as follows

$$\begin{aligned} I = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(f(v)^{\frac{p}{r}} g(v-u)^{\frac{q}{r}}\right) \left(f(v)^{\frac{p}{q'}}\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)^{\frac{r'}{q'}}\right) \\ \left(g(v-u)^{\frac{q}{p'}}\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)^{\frac{r'}{p'}}\right)\mathrm{d}u\,\mathrm{d}v \ \leq \ I_1\, I_2\, I_3\,, \end{aligned} \tag{4.23}$$

where

$$I_1 := \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)^p g(v-u)^q \,\mathrm{d}u\,\mathrm{d}v\right)^{\frac{1}{r}}$$

$$I_2 := \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)^p\, \mathcal{P}(\tau_v \mathcal{R}\psi, 1)(u)^{r'} \mathrm{d}u\,\mathrm{d}v\right)^{\frac{1}{q'}}$$

$$I_3 := \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(v-u)^q\, \mathcal{P}(\tau_v \mathcal{R}\psi, 1)(u)^{r'} \mathrm{d}u\,\mathrm{d}v\right)^{\frac{1}{p'}}$$

$$= \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(v)^q \mathcal{P}(1, \tau_{-v}\psi)(u)^{r'} \mathrm{d}u\,\mathrm{d}v\right)^{\frac{1}{p'}}.$$

Recall that τ and $\mathcal{R}$ are unitary operators in the L^p spaces, thus, from (4.23) and Theorem 4.2.3 we obtain

$$I \le C\,\|f\|_{L^p(\mathbb{R}^n)}\,\|g\|_{L^q(\mathbb{R}^n)}\,\|\psi\|_{L^{r'}(\mathbb{R}^n)},$$

with constant given by

$$C = \left|S^{n-2}\right| \left(\int_{-1}^{1} \left(\tfrac{1-s}{2}\right)^{-\frac{n}{2r'}} \mathrm{d}\xi_n^b(s)\right)^{\frac{r'}{q'}} \left(\int_{-1}^{1} \left(\tfrac{1+s}{2}\right)^{-\frac{n}{2r'}} \mathrm{d}\xi_n^b(s)\right)^{\frac{r'}{p'}}, \tag{4.24}$$

which concludes the proof in this case.

In the case where $\alpha + \lambda > 0$, we shall use two additional inequalities. From the energy dissipation we have $|v'|^2 + |v'_*|^2 \le |v|^2 + |v_*|^2$ and thus

$$|v'|^\alpha = |v - u^-|^\alpha \le \left(|v|^2 + |v_*|^2\right)^{\alpha/2} \le 2^{\alpha/2}\left(|v|^\alpha + |v-u|^\alpha\right). \tag{4.25}$$

Also, we shall use

$$|u|^\lambda \le (|v-u| + |v|)^\lambda \le 2^\lambda\left(|v-u|^\lambda + |v|^\lambda\right). \tag{4.26}$$

Let $\psi_\alpha(v) = \psi(v)|v|^\alpha$ and repeat the procedure above for the case $\alpha = \lambda = 0$ using (4.25) and (4.26) to obtain

$$\begin{aligned}\int_{\mathbb{R}^n} Q^+(f,g)(v)\psi_\alpha(v)\,\mathrm{d}v &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)g(v-u)\mathcal{P}(\tau_v\mathcal{R}\psi_\alpha,1)(u)\ |u|^\lambda\,\mathrm{d}u\,\mathrm{d}v\\ &\leq 4\ \ 2^{\alpha/2}\ 2^\lambda\, C\,\|f\|_{L^p_{\alpha+\lambda}(\mathbb{R}^n)}\,\|g\|_{L^q_{\alpha+\lambda}(\mathbb{R}^n)}\,\|\psi\|_{L^{r'}(\mathbb{R}^n)}.\end{aligned}$$

This proves that

$$\left\|Q^+(f,g)(v)|v|^\alpha\right\|_{L^r(\mathbb{R}^n)} \leq 4\ \ 2^{\alpha/2}\ 2^\lambda\, C\,\|f\|_{L^p_{\alpha+\lambda}(\mathbb{R}^n)}\,\|g\|_{L^q_{\alpha+\lambda}(\mathbb{R}^n)}.$$

A similar reasoning provides

$$\left\|Q^+(f,g)(v)\right\|_{L^r(\mathbb{R}^n)} \leq 2\ \ 2^\lambda\, C\,\|f\|_{L^p_{\alpha+\lambda}(\mathbb{R}^n)}\,\|g\|_{L^q_{\alpha+\lambda}(\mathbb{R}^n)}\,,$$

and finally

$$\left\|Q^+(f,g)(v)\right\|_{L^r_\alpha(\mathbb{R}^n)} \leq 2^{1/r}\ 4\ \ 2^{\alpha/2}\ 2^\lambda\, C\,\|f\|_{L^p_{\alpha+\lambda}(\mathbb{R}^n)}\,\|g\|_{L^q_{\alpha+\lambda}(\mathbb{R}^n)}\,, \tag{4.27}$$

with C given in (4.24). This concludes the proof.

4.4 Proof of Theorem 4.1.2 - Hardy-Littlewood-Sobolev Inequality for Soft Potentials

In this section we study the collision operator for soft potentials and prove Theorem 4.1.2. From (4.8) we have

$$\begin{aligned} I := \int_{\mathbb{R}^n} Q^+(f,g)(v)\,\psi(v)\,\mathrm{d}v &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)g(v-u)\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)\,|u|^\lambda\,\mathrm{d}u\,\mathrm{d}v\\ &= \int_{\mathbb{R}^n} f(v)\left(\int_{\mathbb{R}^n}\tau_v\mathcal{R}g(u)\,\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)\ |u|^\lambda\mathrm{d}u\right)\mathrm{d}v. \end{aligned}\tag{4.28}$$

Applying Hölder's inequality and then Theorem 4.2.3 to the inner integral of (4.28) we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \tau_v \mathcal{R} g(u)\, \mathcal{P}(\tau_v \mathcal{R}\psi,1)(u)\ |u|^\lambda \mathrm{d}u \leq\ & \|\mathcal{P}(\tau_v\mathcal{R}\psi,1)\|_{L^a(\mathbb{R}^n,\,\mathrm{d}\nu_\lambda)}\, \|\tau_v\mathcal{R}g\|_{L^{a'}(\mathbb{R}^n,\,\mathrm{d}\nu_\lambda)} \\ &\leq C_1\, \|\tau_v\mathcal{R}\psi\|_{L^a(\mathbb{R}^n,\,\mathrm{d}\nu_\lambda)}\, \|\tau_v\mathcal{R}g\|_{L^{a'}(\mathbb{R}^n,\,\mathrm{d}\nu_\lambda)} \\ &= C_1\Big[\big(|\psi|^a * |u|^\lambda\big)(v)\Big]^{1/a} \Big[\big(|g|^{a'} * |u|^\lambda\big)(v)\Big]^{1/a'},\end{aligned}$$

where $1/a + 1/a' = 1$ (a to be chosen later), and the constant C_1 given by

$$C_1 = \left|S^{n-2}\right| \int_{-1}^{1} \left(\tfrac{1-s}{2}\right)^{-\frac{n+\lambda}{2a}}\ \mathrm{d}\xi_n^b(s)\,.$$

Therefore we obtain

$$I \leq C_1 \int_{\mathbb{R}^n} f(v)\,\Big[\big(|\psi|^a * |u|^\lambda\big)(v)\Big]^{1/a} \Big[\big(|g|^{a'} * |u|^\lambda\big)(v)\Big]^{1/a'} \mathrm{d}v. \tag{4.29}$$

Applying Hölder's inequality in (4.29) with exponents $1/p + 1/b + 1/c = 1$ (b and c to be chosen later) we arrive at

$$I \leq C_1\ \|f\|_{L^p(\mathbb{R}^n)}\ \Big\||\psi|^a * |u|^\lambda\Big\|^{1/a}_{L^{b/a}(\mathbb{R}^n)}\ \Big\||g|^{a'} * |u|^\lambda\Big\|^{1/a'}_{L^{c/a'}(\mathbb{R}^n)} \tag{4.30}$$

We now use the classical Hardy-Littlewood-Sobolev inequality to obtain

$$\Big\||\psi|^a * |u|^\lambda\Big\|_{L^{b/a}(\mathbb{R}^n)} \leq C_2\ \|\psi\|^a_{L^{ad}(\mathbb{R}^n)} \tag{4.31}$$

and

$$\Big\||g|^{a'} * |u|^\lambda\Big\|_{L^{c/a'}(\mathbb{R}^n)} \leq C_3\ \|g\|^{a'}_{L^{a'e}(\mathbb{R}^n)}\,, \tag{4.32}$$

where

$$1 + \frac{a}{b} = \frac{1}{d} - \frac{\lambda}{n} \quad \text{and} \quad 1 + \frac{a'}{c} = \frac{1}{e} - \frac{\lambda}{n}\,.$$

The constants C_2 and C_3 (generally not sharp) are explicit in [47, p. 106]. Finally putting together (4.31) and (4.32) with (4.30) we arrive at

$$I \leq C_1\, C_2^{1/a}\, C_3^{1/a'} \|f\|_{L^p(\mathbb{R}^n)}\, \|g\|_{L^{a'e}(\mathbb{R}^n)}\, \|\psi\|_{L^{ad}(\mathbb{R}^n)}. \tag{4.33}$$

To conclude the proof of the theorem it would suffice to have in (4.33) the relations $a'e = q$ and $ad = r'$. Now it comes the moment to choose our variables. All the inequalities we used above will be well-posed if the following relations are satisfied

$$(*) \begin{cases} \dfrac{1}{a} + \dfrac{1}{a'} = 1, \qquad 1 \leq a \leq \infty \\ \dfrac{1}{p} + \dfrac{1}{b} + \dfrac{1}{c} = 1, \quad 1 < b, c < \infty \\ 1 + \dfrac{a}{b} = \dfrac{1}{d} - \dfrac{\lambda}{n}, \qquad b > a, \qquad 1 < d < \infty \\ 1 + \dfrac{a'}{c} = \dfrac{1}{e} - \dfrac{\lambda}{n}, \qquad c > a', \qquad 1 < e < \infty \\ a'e = q \\ ad = r' \end{cases}$$

The last two equations determine d and e in terms of a. The remaining linear system (in the variables $1/a$, $1/a'$, $1/b$ and $1/c$) in undetermined because of the original relation

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{\lambda}{n} + \frac{1}{r}.$$

One can check that the choice

$$\frac{1}{b} = \frac{1}{r'} - \frac{1}{a}\left(1 + \frac{\lambda}{n}\right)$$

and

$$\frac{1}{c} = \frac{1}{q} - \frac{1}{a'}\left(1 + \frac{\lambda}{n}\right)$$

with any $1/a$ in the non-empty interval

$$\max\left\{\frac{1}{r'(2+\frac{\lambda}{n})}, 1 - \frac{1}{q(1+\frac{\lambda}{n})}\right\} < \frac{1}{a} < \min\left\{\frac{1}{r'(1+\frac{\lambda}{n})}, 1 - \frac{1}{q(2+\frac{\lambda}{n})}\right\}$$

provides a solution for $(*)$.

Chapter 5

Regularity of Maximal Operators

5.1 Maximal Operators in Sobolev Spaces

Over the last decade there has been considerable interest in understanding the regularity properties of maximal and singular integral operators, for instance how the weak differentiability is preserved. The first work in this direction is due to J. Kinnunen [39] in 1997 when he observed that the classical Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{R>0} \frac{1}{m(B_R)} \int_{B_R(x)} |f(y)| \, \mathrm{d}y, \tag{5.1}$$

is bounded on the Sobolev space $W^{1,p}(\mathbb{R}^n)$ for $p > 1$, using functional analytic tools (weak compactness arguments). Only in 2007 was H. Luiro [51] able to prove that the operator $M : W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$, for $p > 1$, is also a continuous operator (observe that continuity cannot be inferred from the boundedness in this case).

For $\Omega \subset \mathbb{R}^n$ a proper open subset of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ we can define the local maximal operator at a point $x \in \Omega$ by

$$M_\Omega f(x) = \sup_{0<R<\delta_x} \frac{1}{m(B_R)} \int_{B_R(x)} |f(y)| \, \mathrm{d}y, \tag{5.2}$$

where the supremum is taken over all radii R such that $0 < R < \delta_x := \operatorname{dist}(x, \partial\Omega)$. The regularity theory was extended to this operator by Kinnunen and Lindqvist in [40], where they proved that $M_\Omega : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ boundedly for $p > 1$, and the continuity was proved by Luiro in [52]. Some related operators were also considered in the literature, for instance, the non-centered maximal operator by H. Tanaka [63] and the fractional maximal operator by Kinnunen and Saksman in [41]. Other interesting papers related to this topic are [4] and [33].

In the first part of this chapter we will extend the regularity theory for the following family of bilinear maximal operators in $\mathbb{R}^n$. For $\alpha \neq 1$ define

$$\begin{aligned} \mathcal{M}(f,g)(x) &= \sup_{R>0} \frac{1}{m(B_R)} \int_{B_R} |f(x-\alpha y)g(x-y)| \, \mathrm{d}y \\ &:= \sup_{R>0} \fint_{B_R} |f(x-\alpha y)g(x-y)| \, \mathrm{d}y, \end{aligned} \tag{5.3}$$

where B_R is the ball of radius R centered at the origin, and $m(A)$ denotes the n-dimensional Lebesgue measure of the measurable set $A \subset \mathbb{R}^n$. An application of Hölder's inequality tells us that this operator maps $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$ where $1/p+1/q = 1/r$, $1 < p, q < \infty$ and $r > 1$. In 2000, M. Lacey [43] showed that the family of one-dimensional bilinear maximal operators defined by (5.3) maps $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ into $L^1(\mathbb{R})$ where $1/p + 1/q = 1$, $1 < p, q < \infty$, solving a conjecture posed by A. Calderón in 1964. We shall prove here the following result.

Theorem 5.1.1. *Given $\alpha \neq 1$, the bilinear maximal operator $\mathcal{M}$ defined in* (5.3) *maps $W^{1,p}(\mathbb{R}^n) \times W^{1,q}(\mathbb{R}^n) \to W^{1,r}(\mathbb{R}^n)$ boundedly, where $1/p + 1/q =$*

$1/r$, $1 < p, q < \infty$ *and*

(a) $r \geq 1$, *if* $n = 1$;

(b) $r > 1$, *if* $n > 1$.

Boundedness is a consequence of the following pointwise estimate

$$|\nabla \mathcal{M}(f,g)(x)| \leq \mathcal{M}(f, |\nabla g|)(x) + \mathcal{M}(|\nabla f|, g)(x) \quad a.e. \ x \in \mathbb{R}^n. \tag{5.4}$$

Because of Lacey's theorem, the case $n = 1$ and $r = 1$ becomes the key difference between the bilinear maximal and the prior works on the classical Hardy-Littlewood maximal operator. The functional analytic arguments in [39] and [40], relying on the reflexivity of $L^r(\mathbb{R}^n)$ for $r > 1$, are no longer available. To overcome this difficult, we adopt here the geometric measure theory approach introduced by Hajłasz and Onninen in [33]. By an adaptation of the argument of Luiro [51], one can also prove that $\mathcal{M} : W^{1,p}(\mathbb{R}^n) \times W^{1,q}(\mathbb{R}^n) \to W^{1,r}(\mathbb{R}^n)$ continuously. The details for this part can be checked in [17].

Remark 5.1.1. It is believed that the bilinear maximal operator in $\mathbb{R}^n$, $n > 1$, also maps $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ if $1/p + 1/q = 1$, $1 < p, q < \infty$ (M. Lacey, personal communication). If this is indeed the case, we can include $r = 1$, $n > 1$ in Theorem 5.1.1 above with our proof.

Throughout this chapter we consider the following norm for $f \in W^{1,p}$

$$||f||_{1,p} = ||f||_p + ||\nabla f||_p \ ,$$

where ∇f denotes the weak gradient of the Sobolev function f.

5.2 Proof of Theorem 5.1.1 - Regularity of the Bilinear Maximal Operator

The proof of Theorem 5.1.1 is independent of the parameter α (as long as $\alpha \neq 1$ to guarantee Lacey's theorem) and henceforth we work with $\alpha = -1$. Let $f \in W^{1,p}(\mathbb{R}^n)$ and $g \in W^{1,q}(\mathbb{R}^n)$. Since $|f| \in W^{1,p}(\mathbb{R}^n)$ and $|\nabla|f|| = |\nabla f|$ we can assume that f and g are non-negative.

We start with $f, g \in C_0^\infty(\mathbb{R}^n)$ and fix $x, y \in \mathbb{R}^n$. We may assume that $\mathcal{M}(f,g)(x) \geq \mathcal{M}(f,g)(y)$. Let us take a sequence of radii $\{r_n\}_{n=1}^\infty$, $0 < r_n < \infty$ such that

$$\lim_{n\to\infty} \fint_{B_{r_n}} f(x+z)g(x-z)\,\mathrm{d}z = \mathcal{M}(f,g)(x)\,,$$

and write

$$u_{r_n}(x) = \fint_{B_{r_n}} f(x+z)g(x-z)\,\mathrm{d}z$$

for all $n \in \mathbb{N}$. Since

$$|\mathcal{M}(f,g)(x) - \mathcal{M}(f,g)(y)| \;\leq\; (\mathcal{M}(f,g)(x) - u_{r_n}(x)) + (u_{r_n}(x) - u_{r_n}(y))$$

for all $n \in \mathbb{N}$, we have

$$|\mathcal{M}(f,g)(x) - \mathcal{M}(f,g)(y)| \;\leq\; \limsup_{n\to\infty}\,(u_{r_n}(x) - u_{r_n}(y))\,. \tag{5.5}$$

By combining equation (5.5) with the following estimate

$$
\begin{aligned}
& |u_{r_n}(x) - u_{r_n}(y)| = \\
= \ & \left| \fint_{B_{r_n}} \Big\{ f(x+z)g(x-z) - f(y+z)g(y-z) \Big\} \, \mathrm{d}z \right| \\
= \ & \left| \fint_{B_{r_n}} \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \Big\{ f(tx+(1-t)y+z) g(tx+(1-t)y-z) \Big\} \, \mathrm{d}t \, \mathrm{d}z \right| \\
\leq \ & |y-x| \int_0^1 \fint_{B_{r_n}} \Big\{ |\nabla f(tx+(1-t)y+z)||g(tx+(1-t)y-z)| \\
& \qquad + \ |f(tx+(1-t)y+z)||\nabla g(tx+(1-t)y-z)| \Big\} \, \mathrm{d}z \, \mathrm{d}t \\
\leq \ & |y-x| \int_0^1 \Big\{ \mathcal{M}(|\nabla f|, g)(tx+(1-t)y) + \mathcal{M}(f, |\nabla g|)(tx+(1-t)y) \Big\} \, \mathrm{d}t \\
= \ & \int_{\overline{xy}} \Big\{ \mathcal{M}(|\nabla f|, g) + \mathcal{M}(f, |\nabla g|) \Big\} \, \mathrm{d}\mathcal{H}^1,
\end{aligned}
$$

we obtain

$$
|\mathcal{M}(f,g)(x) - \mathcal{M}(f,g)(y)| \leq \int_{\overline{xy}} \Big\{ \mathcal{M}(f, |\nabla g|) + \mathcal{M}(|\nabla f|, g) \Big\} \, \mathrm{d}\mathcal{H}^1 \tag{5.6}
$$

for all $x, y \in \mathbb{R}^n$.

Now consider $f \in W^{1,p}(\mathbb{R}^n)$ and $g \in W^{1,q}(\mathbb{R}^n)$. Fix a vector $\nu \in S^{n-1}$ and consider sequences $\{f_j\}_{j=1}^\infty$ and $\{g_j\}_{j=1}^\infty$ of functions in $C_0^\infty(\mathbb{R}^n)$ such that

$$
f_j \to f \ \text{ in } \ W^{1,p}(\mathbb{R}^n) \quad \text{ and } \quad g_j \to g \ \text{ in } \ W^{1,q}(\mathbb{R}^n).
$$

From the continuity of the bilinear maximal operator in $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ for $1/p + 1/q = 1/r$, $1 < p, q < \infty$ and $r > 1$ (here we can also include

$n = 1$ and $r = 1$) we have

$$\mathcal{M}(f_j, g_j) \rightarrow \mathcal{M}(f, g) \quad \text{in} \quad L^r(\mathbb{R}^n), \tag{5.7}$$

$$\mathcal{M}(f_j, |\nabla g_j|) \rightarrow \mathcal{M}(f, |\nabla g|) \quad \text{in} \quad L^r(\mathbb{R}^n), \tag{5.8}$$

$$\mathcal{M}(|\nabla f_j|, g_j) \rightarrow \mathcal{M}(|\nabla f|, g) \quad \text{in} \quad L^r(\mathbb{R}^n). \tag{5.9}$$

Using the fact that if $h_j \rightarrow h$ in $L^r(\mathbb{R}^n)$ then there is a subsequence such that for almost all lines l parallel to ν the restriction of h_j to l converge in $L^r(l)$ to the restriction of h to l, a standard approximation argument based on (5.6)-(5.9) gives

$$|\mathcal{M}(f, g)(x) - \mathcal{M}(f, g)(y)| \leq \int_{\overline{xy}} \Big\{ \mathcal{M}(f, |\nabla g|) + \mathcal{M}(|\nabla f|, g) \Big\} \, \mathrm{d}\mathcal{H}^1$$

almost everywhere on almost all lines parallel to ν. This is sufficient to conclude that the weak derivative in the ν-direction $D_\nu \mathcal{M}(f, g)(x)$ exists for almost every $x \in \mathbb{R}^n$ (cf. [24, section 4.9]) and satisfies

$$|D_\nu \mathcal{M}(f, g)(x)| \leq \mathcal{M}(f, |\nabla g|)(x) + \mathcal{M}(|\nabla f|, g)(x).$$

Finally, taking the supremum over a countable and dense set of directions $\nu \in S^{n-1}$ we obtain

$$|\nabla \mathcal{M}(f, g)(x)| \leq \mathcal{M}(f, |\nabla g|)(x) + \mathcal{M}(|\nabla f|, g)(x)$$

for almost every $x \in \mathbb{R}^n$ which establishes (5.4).

5.3 Almost Everywhere and Weak Convergence

We now turn our attention to the classical Hardy-Littlewood maximal operator to add some remarks to its regularity theory from a different perspective. Both the global and the local maximal operators defined in (5.1) and (5.2) are known to be bounded from L^p to L^p when $p > 1$. In this case, the sublinearity of the operator implies continuity. As already mentioned earlier in this chapter, the operator M defined in (5.1) is bounded and continuous from $W^{1,p}(\mathbb{R}^n)$ to $W^{1,p}(\mathbb{R}^n)$ (see [39] and [51]) and the local maximal operator M_Ω is bounded and continuous from $W^{1,p}(\Omega)$ to $W^{1,p}(\Omega)$ (see [40] and [52]).

We may ask ourselves if these classical maximal operators preserve other types of convergence, for instance pointwise convergence almost everywhere or weak convergence . The goal of this section is to settle the discussion about these issues providing counterexamples and positive results on this direction.

Proposition 5.3.1. *The maximal operators $M : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ and M_Ω : $L^p(\Omega) \to L^p(\Omega)$, for $p > 1$, do not preserve pointwise convergence almost everywhere.*

Proof. This follows simply from the observation that

$$M(f)(x) \geq C_f |x|^{-n} \quad \text{whenever } |x| \geq 1,$$

where $C_f := \dfrac{||f||_{L^1(B_1)}}{2^n \omega_n}$. Here ω_n is the volume of the unit ball $B_1 \subset \mathbb{R}^n$. We consider the sequence

$$u_k(x) = \frac{1}{m(B_{\frac{1}{k}})}\chi_{B_{\frac{1}{k}}}(x).$$

Clearly, $u_k \to 0$ a.e. but $M(u_k) \not\to 0$ a.e.. The argument for the local case is just a simple adaptation of this one. $\square$

Issues about the stability of the weak convergence under nonlinear operators are much more interesting and have been studied in [55] for a certain class called Nemytskii nonlinearities with applications to differential equations in [66]. Given an operator $T : E \to F$ between Banach spaces and $u_k \rightharpoonup u$ in E, the question is whether or not we have $T(u_k) \rightharpoonup T(u)$ in F (in the affirmative case for all such sequences $\{u_k\}_{k\geq 1}$, we say that T is sequentially weakly continuous). We show below a counterexample in this direction.

Proposition 5.3.2. *The maximal operators $M : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ and $M_\Omega : L^p(\Omega) \to L^p(\Omega)$, for $p > 1$, are not sequentially weakly continuous.*

Proof. We start with the local case. Let $\Omega = (-1, 1) \subset \mathbb{R}$ and consider the orthonormal system in $L^2(-1, 1)$ given by $u_n(x) = \sin(2\pi nx)$, $n = 1, 2, 3, ...$ Therefore we have $u_n \rightharpoonup 0$ in $L^2(-1, 1)$ but we claim that $M_\Omega(u_n) \not\rightharpoonup 0$ in

$L^2(-1,1)$. To see this, let us fix a radius $r < \frac{1}{2}$ and consider the inner product

$$\begin{aligned}
\langle 1, M_\Omega(u_n)\rangle_{L^2(-1,1)} &= \int_{-1}^{1} M_\Omega(u_n)(x)\,\mathrm{d}x \geq \int_{-\frac{1}{2}}^{\frac{1}{1}} M_\Omega(u_n)(x)\,\mathrm{d}x \\
&\geq \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\fint_{B_r} |u_n(x+y)|\,\mathrm{d}y \right) \mathrm{d}x \\
&= \fint_{B_r} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} |\sin(2\pi n x + 2\pi n y)|\,\mathrm{d}x \right) \mathrm{d}y \\
&= \fint_{B_r} C\,\mathrm{d}y = C,
\end{aligned}$$

where $C > 0$ is a constant. This proves our claim.

For the classical Hardy-Littlewood maximal operator we give the following counterexample in $L^2(\mathbb{R})$:

$$u_n(x) = \frac{\sin(2\pi n x)}{1+x^2}.$$

We have $u_n \rightharpoonup 0$ in $L^2(\mathbb{R})$ as a consequence of the Riemann-Lebesgue lemma, but we claim that $M(u_n) \not\rightharpoonup 0$ in $L^2(\mathbb{R})$. To verify this, fix $0 < r < 1$ and observe that

$$\begin{aligned}
\langle M(u_n), \frac{1}{1+x^2}\rangle_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} M(u_n)(x) \frac{1}{1+x^2}\,\mathrm{d}x \\
&\geq \int_{\mathbb{R}} \left(\fint_{B_r} |u_n(x+y)|\,\mathrm{d}y \right) \frac{1}{1+x^2}\,\mathrm{d}x \\
&= \fint_{B_r} \left(\int_{\mathbb{R}} \frac{|\sin(2\pi n(x+y)|}{(1+(x+y)^2)(1+x^2)}\,\mathrm{d}x \right) \mathrm{d}y \\
&\geq \fint_{B_r} C\,\mathrm{d}y = C,
\end{aligned}$$

where $C > 0$ is a constant. This finishes the proof of our claim. $\square$

It is interesting to compare the previous counterexamples with the following positive results in Sobolev spaces.

Proposition 5.3.3. *Suppose Ω is a bounded domain with Lipschitz boundary. Then, the local maximal operator $M_\Omega : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ is sequentially weakly continuous for $p > 1$.*

Proof. Let $f_j \rightharpoonup f$ in $W^{1,p}(\Omega)$. Since $M_\Omega : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ is a bounded operator, the sequence $M_\Omega(f_j)$ must admit a weakly convergent subsequence, by reflexivity. This way, we can assume

$$M_\Omega(f_j) \rightharpoonup g \;\text{ in }\; W^{1,p}(\Omega).$$

By the compactness of the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ and the continuity in $L^p(\Omega)$ of the local maximal operator, we have

$$M_\Omega(f_j) \to M_\Omega(f) \;\text{ in }\; L^p(\Omega).$$

In particular, $M_\Omega(f) = g$ and this finishes the proof. $\square$

Theorem 5.3.4. *Let $1 < p < \infty$ and suppose $u_k \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^n)$. There exists a subsequence $M(u_{k_j}) \to M(u)$ a.e. in $\mathbb{R}^n$.*

Proof. By the sublinearity of the maximal operator, it is enough to prove the case where $u \equiv 0$. Let us consider $B = B_L(0)$, where $L > 0$.

First, we observe that if $f \in L^p(\mathbb{R}^n)$, there exists a universal $C > 0$ (depending only on the dimension n) such that

$$⨍_{B_R(x)} |f(y)|\, \mathrm{d}y \leq C\, R^{-\frac{n}{p}}\, ||f||_{L^p(\mathbb{R}^n)}.$$

For each $m = 1, 2, 3, \ldots$ we can take $R_m > 0$ large enough so that, for every $x \in \mathbb{R}^n$,

$$\fint_{B_R(x)} |u_k(y)| \,\mathrm{d}y \leq \frac{1}{m} \quad \text{for all } k \in \mathbb{N} \quad \text{whenever } R \geq R_m. \tag{5.10}$$

Let us consider now $B_m^\star := B_{L+2R_m}(0)$ and the local maximal operator with respect to this ball. For any $x \in B$, if $\delta_x = \operatorname{dist}(x, \partial B_m^\star)$, we have by the estimate (5.10)

$$\begin{aligned} M(u_k)(x) &= \max\Big\{ \sup_{0<R<\delta_x} \fint_{B_R(x)} |u_k(y)| \,\mathrm{d}y,\ \sup_{R>\delta_x} \fint_{B_R(x)} |u_k(y)| \,\mathrm{d}y \Big\} \\ &\leq \max\Big\{ M_{B_m^\star}(u_k)(x),\ \frac{1}{m} \Big\}. \end{aligned} \tag{5.11}$$

Since $W^{1,p}(\mathbb{R}^n) \hookrightarrow W^{1,p}(B_m^\star)$ continuously and $W^{1,p}(B_m^\star) \hookrightarrow L^p(B_m^\star)$ compactly, by the continuity of the local maximal operator $M_{B_m^\star}$, $M_{B_m^\star}(u_k) \to 0$ in $L^p(B_m^\star)$. Therefore, there is a subsequence $M_{B_m^\star}(u_{k_j}^m) \to 0$ a.e. in $B_m^\star$. From (5.11) we conclude that

$$\limsup_{j\to\infty} M(u_{k_j}^m)(x) \leq \frac{1}{m} \quad \text{a.e. in } B.$$

Using the Cantor diagonal argument we can find a subsequence $\{u_{k_j}\}$ such that

$$M(u_{k_j})(x) \to 0 \quad \text{a.e. in } B.$$

Since the original ball B was arbitrary, we can use once more the Cantor diagonal argument applied to $\mathbb{R}^n = \bigcup_{n=0}^{\infty} B_n(0)$ to conclude the proof. $\square$

Corollary 5.3.5. *Assume* $1 < p < \infty$. *The maximal operator* $M : W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ *is sequentially weakly continuous.*

Proof. The proof is similar to the proof for the local maximal operator given in Proposition 5.3.3, with the help of the previous theorem. Let $u_k \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^n)$. By the boundedness of the maximal operator in $W^{1,p}(\mathbb{R}^n)$, we can assume $M(u_k) \rightharpoonup g$ in $W^{1,p}(\mathbb{R}^n)$. By the previous theorem, there exists a subsequence $M(u_{k_j}) \to M(u)$ a.e in $\mathbb{R}^n$. This is sufficient to conclude that $M(u) = g$. $\square$

We observe that Theorem 5.3.4 is optimal in the sense that one cannot replace the weak convergence in $W^{1,p}(\mathbb{R}^n)$ for weak convergence in $L^p(\mathbb{R}^n)$. Proposition 5.3.2 above presents a sequence $u_k \rightharpoonup 0$ in $L^2(\mathbb{R})$ such that $M(u_k) \not\to 0$ a.e.

Finally, we point out that Theorem 5.3.4 and its corollary are also optimal in the right hand side. We present an example showing that the maximal operator is not compact in the sense that it does not map weakly convergent sequences into strongly convergent sequences.

Proposition 5.3.6. *For $1 < p < \infty$, the maximal operators $M : W^{1,p}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ and $M_\Omega : W^{1,p}(\Omega) \to L^p(\Omega)$ are not compact.*

Proof. For the local case, consider the sequence of disjoint balls $B_k := B_{1/2}(ke_1)$, where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$, put $\Omega = \bigcup_{k=1}^{\infty} B_k$ and take the sequence of functions $u_k := m(B_k)^{-1/p}\chi_{B_k} \in C^\infty(\Omega)$. For the global case let $u \in C_0^\infty(\mathbb{R}^n)$ and consider the sequence $u_k(x) = u(x - k)$. $\square$

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